Nuclear and Particle Physics

Barem

i)The Heisenberg Uncertainty Principle relates the uncertainties ΔE in energy and Δt in time by:

$$\Delta E\, \Delta t \geq \frac{h}{4\pi}$$

$d \approx c\Delta t$, where c is the speed of light as the maximum value hypothetical possible.

$$\Delta t \approx \frac{d}{c} = \frac{10^{-15} m}{3 \times 10^{8} m/sec} \approx 3.3 \times 10^{-24} sec$$

$$\Delta E \approx \frac{h}{4\pi \Delta t} = \frac{6{,}63 \times 10^{-34} Jsec}{4 \times 3{,}14 \times 3{,}3 \times 10^{-24} sec} = 1{,}6 \times 10^{-11} J = 1{,}6 \times 10^{-11} J \times \frac{1\, MeV}{1{,}6 \times 10^{-13} J} = 100\, MeV$$

$\Delta E = mc^2$ and thus $m_\pi = \frac{\Delta E}{c^2} = 100\, MeV/c^2$.

Punctaj: **2p**

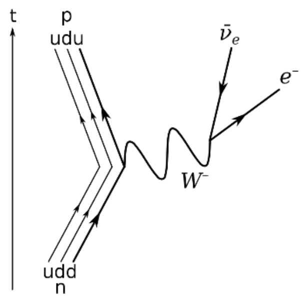
ii) The pion must then be captured and, thus, cannot be directly observed because that would amount to a permanent violation of mass-energy conservation. Such particles (like **the pion it is a virtual particle, because they cannot be directly observed but their *effects* can be directly observed**.)

Punctaj: **1p**

iii) The correct result and numerical value is: $E_K = \frac{(m_p + m_n + m_\pi)^2 c^2 - (m_p + m_n)^2 c^2}{2m_n} \cong 280\, MeV$

Punctaj: **2p**

iv)



Punctaj: **2p**

v) What was the dominant interaction (force) in this case. Calculate the radius of action of this interaction in the considered disintegration.

$$\Delta E\,\Delta t \geq \frac{h}{4\pi}$$

$$x \approx c\Delta t = \frac{h}{4\pi\,\Delta E} = \frac{h}{4\pi m c^2} = \frac{3\times 10^8\,\frac{m}{sec}\times 6,63\times 10^{-34}\,Jsec}{4\times 3,14\times\frac{83\,GeV}{c^2}\times c^2\times 1,6\times 10^{-10}\,\frac{J}{GeV}} = 1,2\times 10^{-18}\,m$$

Punctaj: **2p**

Oficiu: **1p**

Total: **10p**

Thermodynamics

1. The state equation of a thermodynamic system is:

$$p = \frac{AT^3}{V}$$

in which p, V and T represent pressure, volume and temperature, whereas A is a constant. The expression of the internal energy of the system is provided by the relation:

$$U = BT^n \ln\left(\frac{V}{V_0}\right) + f(T)$$

in which B, n and V_0 are constants, whereas f(T) is a function which depends only on temperature. Find the values of B and n.

Solution

From the fundamental equation of thermodynamics, one obtains:

(1) $$dS = \frac{dU}{T} + \frac{pdV}{T}$$

U is function of state and has total exact differential:

(2) $$U(T,V) = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

(3) $$\left(\frac{\partial U}{\partial T}\right)_V = nBT^{n-1} ln\left(\frac{V}{V_0}\right) + \left(\frac{\partial f}{\partial T}\right)_V$$

(4) $$\left(\frac{\partial U}{\partial V}\right)_T = B\frac{T^n}{V}$$

From the given equation of state:

(5) $$\frac{p}{T} = \frac{AT^2}{V}$$

From (1), (3), (4) and (5), one obtains:

(6) $$dS = \left[\frac{1}{T}\left(nBT^{n-1} ln\left(\frac{V}{V_0}\right) + \left(\frac{\partial f}{\partial T}\right)_V\right)\right] dT + \left[\frac{1}{T}\frac{BT^n}{V} + \frac{AT^2}{V}\right] dV$$

(7) $$dS = \left[nBT^{n-2} ln\left(\frac{V}{V_0}\right) + \frac{1}{T}\left(\frac{\partial f}{\partial T}\right)_V\right] dT + \left[\frac{BT^{n-1} + AT^2}{V}\right] dV$$

S is function of state (S=S(T,V)) and therefore the following relations hold:

(8) $$S(T,V) = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$$

(9) $$\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V}$$

From (7), (8) and (9) one obtains:

(10) $$\frac{\partial\left(nBT^{n-2} ln\left(\frac{V}{V_0}\right)+\frac{1}{T}\left(\frac{\partial f}{\partial T}\right)_V\right)}{\partial V} = \frac{\partial\left(\frac{BT^{n-1}+AT^2}{V}\right)}{\partial T}$$

(11) $$BT^{n-2} = 2AT$$

Therefore:

N=3 and B=2A

Electricity and Magnetism
Grading

Ex officio: 1point

The flux density $B(x)$ in a close region of x-axis is

$$B(x) \cong \frac{\mu_0 I}{2R} \frac{1}{\left[1+(x/R)^2\right]^{3/2}}, \qquad \textbf{0.5p}$$

where $x=0$ is the center of the circular coil.

For small radial deviations $r << R$, $B(x,r) \cong B(x)$ and the magnetic flux through the cross-circular area πr^2 is given by the loop integral $\oint \mathbf{A} \cdot \mathbf{dl}$ of the magnetic vector potential **A**. Since the contribution to this loop integral is only due to the angular component $A_\varphi(x,r)$, we get

$$A_\varphi(x,r) \cong \frac{\mu_0 I R^2}{4} \frac{r}{\left(x^2+R^2\right)^{3/2}} . \qquad \textbf{2p}$$

The corresponding angular component Π_φ of canonical momentum is given by

$$\Pi_\varphi \cong m r^2 \dot{\varphi} - e A_\varphi(x,r) r . \qquad \textbf{3p}$$

Noteworthy, Π_φ is a conserved quantity and since initially **v** is parallel to x-axis, $\Pi_\varphi = 0$ (as follows from $A_\varphi(x,r) \to 0$ for $x \to \pm\infty$). Therefore, we get

$$m r \dot{\varphi} \cong e A_\varphi(r), \qquad \textbf{1p}$$

$$\dot{\varphi} \cong \frac{e A_\varphi(r)}{mr} = \frac{\mu_0 e I R^2}{4m} \frac{1}{\left(x^2+R^2\right)^{3/2}} = \frac{d\varphi}{dx} \cdot \frac{dx}{dt} \cong \frac{d\varphi}{dx} v, \qquad \textbf{1p}$$

$$d\varphi \cong \frac{\mu_0 e I R^2}{4mv} \frac{1}{\left(x^2+R^2\right)^{3/2}} dx . \qquad \textbf{0.5p}$$

Long after passing through the coil, the electron will have rotated around x-axis with an angle $\varphi(I,v)$ given by the fair approximation

$$\varphi \cong \frac{\mu_0 e I R^2}{4mv} \int_{-\infty}^{+\infty} \frac{1}{\left(x^2+R^2\right)^{3/2}} dx = \frac{\mu_0 e I}{2mv} . \qquad \textbf{1p}$$

Total: 10 points

Note: There is also a less elegant method to approach the problem. In essence, it proceeds as follows:

$$B_x(x+dx, r=0) - B_x(x, r=0) = \left(\frac{dB_x}{dx}\right) dx$$

The infinitesimal flux through a closed cylindrical surface centered on x-axis, with length dx and radius $r<<R$, is $d\Phi \cong \pi r^2\left[B_x(x+dx)-B_x(x)\right]-2\pi r B_r(x,r)dx$. Since $\operatorname{div}\mathbf{B}=0$, $d\Phi=0$, hence

$$r\left(\frac{dB_x}{dx}\right)\cong 2B_r(x,r),$$

$$-\frac{3\mu_0 IR^2}{2}rx\left(x^2+R^2\right)^{5/2}=2B_r(x,r).$$

The Lorentz force acting on electron is

$$F_\varphi=evB_r(x,r),$$

or

$$F_\varphi=\frac{d}{dt}(mv_\varphi)=\frac{d}{dt}(m\dot{\varphi}r)=\frac{3\mu_0 eIR^2v}{4}\frac{rx}{\left(x^2+R^2\right)^{5/2}}.$$

Therefore, with $dx=vdt$, we get the same equation as before.

$$\dot{\varphi}\cong\frac{eA_\varphi(r)}{mr}=\frac{\mu_0 eIR^2}{4m}\frac{1}{\left(x^2+R^2\right)^{3/2}},\ \varphi\cong\frac{\mu_0 eI}{2mv}$$

Regardless of the method used, the students received the appropriate score.

Optică - Barem

A. The constructive interference condition for λ′=640 nm is given by:

$$2nl\cos\theta_r+\frac{\lambda'}{2}=m'\lambda' \qquad 0.25\ p$$

while the destructive interference condition for λ′′=400 nm is given by:

$$2nl\cos\theta_r+\frac{\lambda''}{2}=\left(m''+\frac{1}{2}\right)\lambda'' \qquad 0.25\ p$$

where m', m'' are positive integer numbers, θ_r is the refraction angle for the respective incidence angle $i=30^o$, $\sin i=n\sin\theta_r$. Considering the refraction law (Snell's law) and rearranging the first two equations may be written as:

$$2l\sqrt{n^2-\sin^2 i}=\frac{2m'-1}{2}\lambda',$$

$$2l\sqrt{n^2-\sin^2 i}=m''\lambda'' \qquad 0.25\ p$$

By combining the two equations, we obtain:

$$\frac{2m''}{2m'-1}=\frac{\lambda'}{\lambda''}=\frac{8}{5}\ sau\ 5m''=4\left(2m'-1\right) \qquad 0.25\ p$$

m′′=4m si m′=(5m+1)/2 cu m=1,3,5…The minimum thickness is then obtained for m′′=4, m′=3: 0.5 p

$$l=\frac{2\lambda''}{\sqrt{n^2-\sin^2 i}}\quad l=0.65\ \mu m \qquad 0.5\ p$$

B. At the air-layer separation surface (n_o-n_1):

$$\begin{cases} E_o+E_r=E_{1t}+E_{1r} \\ n_oE_o-n_oE_r=n_1E_{1t}-n_1E_{1r} \end{cases} \quad (1) \qquad 0.5\ p$$

And at the layer-lens separation surface (n_1-n):

$$\begin{cases} E_{1t}e^{-ikl}+E_{1r}e^{ikl}=E_t \\ n_1E_{1t}e^{-ikl}-n_1E_{1r}e^{ikl}=nE_t \end{cases} \quad \begin{cases} E_{1t}\frac{n+n_1}{2n_1}E_te^{ikl} \\ E_{1r}=\frac{n_1-n}{2n_1}E_te^{-ikl} \end{cases} \qquad 0.75\ p$$

$$\begin{cases} E_o+E_r=\frac{n+n_1}{2n_1}E_te^{ikl}+\frac{n_1-n}{2n_1}E_te^{-ikl} \\ n_oE_o-n_oE_r=\frac{n+n_1}{2}E_te^{ikl}-\frac{n_1-n}{2}E_te^{-ikl} \end{cases}$$

$$\begin{cases} E_o+E_r=\left(coskl+i\frac{n}{n_1}sinkl\right)E_t \\ n_o E_o - n_o E_r=\left(i n_1 sinkl+n\, coskl\right)E_t \end{cases}$$

The reflection coefficient in amplitude can be obtained from the previous equations as:

$$r=\frac{E_r}{E_o}=\frac{n_1\left(n_o-n\right)coskl+i\left(n n_o-n_1^2\right)sinkl}{n_1\left(n_o+n\right)coskl+i\left(n n_o+n_1^2\right)sinkl}$$ 0.5 p

The equation of the energy reflection coefficient is then:

$$R=r r^{\square}=\frac{\left[n_1\left(n_o-n\right)coskl\right]^2+\left[\left(n n_o-n_1^2\right)sinkl\right]^2}{\left[n_1\left(n_o+n\right)coskl\right]^2+\left[\left(n n_o+n_1^2\right)sinkl\right]^2}$$ 0.25 p

The layer is anti-reflective, i.e. R=0, if $nn_o=n_1^2$ and *coskl=0, kl=π/2:* 0.5 p

$n_1=\sqrt{n}=1.22$ 0.25 p

$l=\frac{\lambda_1}{4}=\frac{\lambda_o}{4 n_1}=102 nm$ 0.25 p

C. The equations between the incident and the reflected components for two consecutive separating surfaces can be expressed as (1):

Or using matrices as :

$$\begin{pmatrix} E_o+E_r \\ n_o E_o-n_o E_r \end{pmatrix}=\begin{pmatrix} coskl & \frac{i}{n_1}sinkl \\ i n_1 sinkl & coskl \end{pmatrix}\begin{pmatrix} E_{1t}+E_{1r} \\ n_1 E_t-n_1 E_{1r} \end{pmatrix}=M\begin{pmatrix} E_{1t}+E_{1r} \\ n_1 E_t-n_1 E_{1r} \end{pmatrix} 0.5 p$$

Where the matrix M is defined for a layer of length l and refractive index n_1.

$$M=\begin{pmatrix} coskl & \frac{i}{n_1}sinkl \\ i n_1 sinkl & coskl \end{pmatrix}$$

$$\begin{pmatrix} E_o+E_r \\ n_o E_o-n_o E_r \end{pmatrix}=\begin{pmatrix} coskl & \frac{i}{n_1}sinkl \\ i n_1 sinkl & coskl \end{pmatrix}\begin{pmatrix} E_t \\ n E_t \end{pmatrix}=\begin{pmatrix} coskl & \frac{i}{n_1}sinkl \\ i n_1 sinkl & coskl \end{pmatrix}\begin{pmatrix} E_t+0 \\ n E_t-n x 0 \end{pmatrix}$$

From this, it results that after transversing N thin layers, we can write the following for the electric field components: $\begin{pmatrix} E_o+E_r \\ n_o E_o-n_o E_r \end{pmatrix}=M_1 M_2 \ldots\ldots M_N\begin{pmatrix} E_t \\ n E_t \end{pmatrix}$

0.25 p

When depositing 2N layers successively onto the surface of the support substrate n, if we have identical odd layers represented by n_1 , l_1 , and also identical odd layers represented by n_2, l_2, then a matrix relationship between the electric fields will result as:

$$\begin{pmatrix} E_o+E_r \\ n_o E_o - n_o E_r \end{pmatrix} = M_1 M_2 \ldots\ldots M_{2N-1} M_{2N} \begin{pmatrix} E_t \\ n E_t \end{pmatrix} = (M_1 M_2)^N \begin{pmatrix} E_t \\ n E_t \end{pmatrix} \qquad 0.25 \text{ p}$$

Where the matrix:

$$M_1 M_2 = \begin{pmatrix} \cos k_1 l_1 & \frac{i}{n_1}\sin k_1 l_1 \\ i n_1 \sin k_1 l_1 & \cos k_1 l_1 \end{pmatrix} \begin{pmatrix} \cos k_2 l_2 & \frac{i}{n_2}\sin k_2 l_2 \\ i n_2 \sin k_2 l_2 & \cos k_2 l_2 \end{pmatrix}$$

This matrix becomes, in the case of some layers, a quarter wave $l_1=\lambda_o/4n_1$, $l_2=\lambda_o/4n_2$:

$$M_1 M_2 = \begin{pmatrix} 0 & \frac{i}{n_1} \\ i n_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{i}{n_2} \\ i n_2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{-n_2}{n_1} & 0 \\ 0 & \frac{-n_1}{n_2} \end{pmatrix} \qquad 0.5 \text{ p}$$

$$\begin{pmatrix} E_o+E_r \\ n_o E_o - n_o E_r \end{pmatrix} = \begin{pmatrix} \left(\frac{-n_2}{n_1}\right)^N & 0 \\ 0 & \left(\frac{-n_1}{n_2}\right)^N \end{pmatrix} \begin{pmatrix} E_t \\ n E_t \end{pmatrix}$$

The reflection coefficient for a mirror with N double layers is:

$$r = \frac{\left(\frac{n_2}{n_1}\right)^{2N} - \frac{n}{n_o}}{\left(\frac{n_2}{n_1}\right)^{2N} + \frac{n}{n_o}} \quad R = \left(\frac{\left(\frac{n_2}{n_1}\right)^{2N} - \frac{n}{n_o}}{\left(\frac{n_2}{n_1}\right)^{2N} + \frac{n}{n_o}} \right)^2 \qquad 0.5 \text{ p}$$

It is observable that the reflection coefficient's magnitude escalates as the number of layers increases, R→1.

For N=1 the anti-reflective layer condition is:

$$\left(\frac{n_2}{n_1}\right)^2 = \frac{n}{n_o} \qquad 0.5 \text{ p}$$

The condition is similar to the one obtained at point b, only this time it is easier to find two materials that have the relative index $n_{21}=\sqrt{n/n_o}$ than to find a material with this index. 0.5 p

The reflection coefficient for a single titanium dioxide-magnesium fluoride double layer is:

$$R = \left(\frac{{n_{21}}^2 - 1}{{n_{21}}^2 + 1} \right)^2 = 25\%$$ 1 p

The number of double layers necessary to obtain a reflection coefficient R is:

$$\left(\frac{n_2}{n_1} \right)^{2N} = \frac{1 + \sqrt{R}}{1 - \sqrt{R}} \; N = \frac{1}{2} \frac{lg \frac{1 + \sqrt{R}}{1 - \sqrt{R}}}{lg \frac{n_2}{n_1}} \; N = 4 \, for \; R = 95\%$$ 1 p

PLANCKS - Conductivity in disordered electronic systems

- Correction key -

2.i)

$$\sum_\alpha F_\alpha = \underset{\substack{\uparrow \\ \text{spin}}}{2} V \int d\epsilon \; n(\epsilon) F(\epsilon) \quad , \quad (\forall) F$$

$$T = 0\,K \Rightarrow f(\epsilon) = \theta(\epsilon_F - \epsilon) \quad \text{(Heaviside function).}$$

Using above eqs.:

$$\mathrm{Re}\,\sigma(\omega) = \frac{2\pi e^2 V}{m^2} \int_{\epsilon_F - \hbar\omega}^{\epsilon_F} \frac{|P_{\alpha\beta}|^2 n(\epsilon)\, n(\epsilon + \hbar\omega)}{\hbar\omega} d\epsilon \Bigg|_{\hbar\omega \ll \epsilon_F} \simeq |P_{\alpha\beta}|^2 n^2(\epsilon_F)$$

$$n(\epsilon) = \frac{(2m)^{3/2}}{4\pi^2\hbar^3} \sqrt{E} \quad \text{(all energies with reference to } \epsilon_c\text{).}$$

$$|P_{\alpha\beta}|^2 = \frac{2\hbar^2\pi l}{3V} \; \frac{1}{1 + (k_\alpha - k_\beta)^2 l^2}$$

$$\epsilon_\alpha = \frac{\hbar^2 k_\alpha^2}{2m} \qquad \epsilon_\alpha - \epsilon_\beta = \hbar\omega \ll \epsilon_F \Rightarrow \epsilon_\alpha \simeq \epsilon_\beta \simeq \epsilon_F$$

$$\epsilon_\alpha - \epsilon_\beta = \frac{\hbar^2(k_\alpha^2 - k_\beta^2)}{2m} = \frac{\hbar^2(k_\alpha - k_\beta)(k_\alpha + k_\beta)}{2m}$$

$$\simeq \frac{\hbar^2 k_F}{m}(k_\alpha - k_\beta).$$

$$\Rightarrow \begin{cases} k_\alpha - k_\beta = \dfrac{m\omega}{\hbar} \dfrac{1}{k_F} \,, \quad l = v_F \tau = \dfrac{\hbar k_F}{m}\tau \\[2mm] |P_{\alpha\beta}|^2 = \dfrac{2\hbar^2\pi l}{3V} \dfrac{1}{1+\omega^2\tau^2} \,, \quad k_F = (3\pi^2 n)^{1/3} \\[2mm] \mathrm{Re}\,\sigma(\omega) = \dfrac{ne^2}{m} \dfrac{\tau}{1+\omega^2\tau^2} \quad \text{(Drude).} \end{cases}$$

Only charge carriers occupying states in the vicinity of Fermi energy ϵ_F, with $\hbar\omega \ll \epsilon_F$, involved in transport.

2.ii).

$k_F l \gg 1$ for "good" conductors, limit: $l_{min} \simeq \frac{1}{k_F}$,

For 3D electron systems: $k_F = (3\pi^2 n)^{1/3}$

$$\Rightarrow \quad \sigma_{min} = \frac{e^2}{3\pi^2\hbar} \frac{1}{l_{min}}$$

2.iii)

$$H = T + V_a + V_b + \cancel{V_{ab}} \qquad \begin{cases} |\psi_1\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle) & \text{bonding} \\ |\psi_2\rangle = \frac{1}{\sqrt{2}}(|a\rangle - |a\rangle) & \text{anti-bonding} \end{cases}$$

$$H_{a,b} = T + V_{a,b}$$

$$\left. \begin{aligned} H|\psi_1\rangle &= \epsilon_1 |\psi_1\rangle \\ H|\psi_2\rangle &= \epsilon_2 |\psi_2\rangle \end{aligned} \right| \Rightarrow \quad \epsilon_{1,2} = \frac{A \pm B}{1 \pm S} \qquad \begin{aligned} A &= \langle a|H|a\rangle \\ B &= \langle a|H|b\rangle \\ S &= \langle a|b\rangle. \end{aligned}$$

2.iv) if $S \ll 1$ (strongly localized states):

$$\epsilon_1 - \epsilon_2 = 2\langle a|H|b\rangle \begin{cases} = 2(\epsilon_0 S + \langle a|V_a|b\rangle) \\ \simeq 2\langle a|V_a|b\rangle. \\ = 2I = 2I_0 e^{-KR} \end{cases}$$

s-type orbital: $\langle \vec{r}|a\rangle = \frac{K^{3/2}}{\sqrt{\pi}} e^{-K|\vec{r}-\vec{R}|}$.

2.v).

$$\dot{x}_j = \frac{i}{\hbar}[H, x_j] = \frac{i}{\hbar}\left[\frac{p^2}{2m}, x_j\right] = \frac{p_j}{m}$$

$$\Rightarrow \quad \langle \alpha|\vec{p}|\beta\rangle = \frac{m}{i\hbar}(\epsilon_\beta - \epsilon_\alpha)\langle \alpha|\vec{r}|\beta\rangle$$

isotropic system: $x_{\alpha\beta}^2 = \frac{1}{3} r_{\alpha\beta}^2$

2.vi) $\hbar\omega \ll \epsilon_F$

$$\sigma(\omega) = \frac{2\pi e^2 V^{-1}}{3\hbar} n(\epsilon_F)^2 (\hbar\omega)^2 \underbrace{\int |\langle \alpha|r|\beta\rangle|^2 d\vec{r}_a d\vec{r}_b}_{\text{average over configurations (uniformly distributed localized states).}}$$

2.vii)

$$\begin{pmatrix} \vec{r}_a \\ \vec{r}_b \end{pmatrix} \longrightarrow \begin{pmatrix} \vec{r}_a \\ \vec{R} = \vec{r}_a - \vec{r}_b \end{pmatrix}$$ change of variables in integral over configs.

$$\Rightarrow \quad |\vec{r}_{\alpha\beta}| = \frac{R}{2}.$$

2.viii) in the limit $KR \gg 1$ (large distances between centers)

$$|\vec{r}_{\alpha\beta}| \propto R e^{-KR}$$

(Direct calculation:

$$\int e^{-K|\vec{r}-\vec{r}_a|}\, \vec{r}\, e^{-K|\vec{r}-\vec{r}_b|} d\vec{r} \propto R e^{-KR} \Big)$$

2. ix)

Minimum hopping distance: $R \geq R_\omega$

$$2\dot{I} = 2 I_0 e^{-\kappa R_\omega} \leq \hbar\omega \Rightarrow R_\omega = \frac{1}{\kappa} \ln \frac{2\dot{I}_0}{\hbar\omega}$$

if centers too close, $\dot{I}$ increases exponentially, electron hopping less probable.

$$\langle \sigma(\omega) \rangle = \frac{2\pi e^2}{3\hbar V} [n(\epsilon_F)]^2 (\hbar\omega)^2 \int d\vec{r}_a \int d\vec{R} \, |r_{\alpha\beta}|^2$$

$|r_{\alpha\beta}|^2 = \frac{R^2}{4}$, $R \in (R_\omega, R_\omega + \frac{1}{\kappa})$ ← if too large distances, $|r_{\alpha\beta}| \propto R e^{-\kappa R} \to 0$.

$$\Rightarrow \langle \sigma(\omega) \rangle = \text{const} \cdot \omega^2 \ln\left(\frac{\hbar\omega}{2I_0}\right)^4$$

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1 point each item.
1. point for start.

$\Rightarrow$ $\boxed{10 \text{ points}}$

1. The average lifetime of the muon in its proper/rest frame is $2.2 \cdot 10^{-6}s$, its rest mass is 106 MeV.
(a) Assuming that muons travel at 99.8% of the speed of light, show that cosmic radiation muons can indeed be detected on the surface of the Earth. Take $d = 10$km as thickness of Earth's atmosphere and $299.8 \cdot 10^6$m/s the speed of light. The direction of the muon is vertical with respect to the ground.
(b) What is the smallest energy required for muons to hit the ground before they decay, assuming that they are produced at an altitude of 10 km above ground? (here you drop the velocity assumption at point (a)!). The direction of the muon is vertical again.
(c) A circular accelerator has a radius of 50m. How many turns can a muon take on average in this ring before it decays if its energy is kept constant at 1 GeV? Here, take $c = 3 \cdot 10^8 m/s$.

Solution. **(a)** The lifetime τ_0 is measured in the rest frame of the muon. Due to time dilation, the lifetime in the rest frame of the Earth will be

$$\tau = \tau_0 \gamma(v),$$

where v is the velocity of the muon relative to Earth. [15%]
With $v = 0.998c$, one obtains

$$\tau = \frac{\tau_0}{\sqrt{1 - \frac{v^2}{c^2}}} \approx 15.8\tau_0.$$

During this time, the muon will move the distance

$$\tau v \approx 15.8 \cdot 2.2 \cdot 10^{-6} \cdot 0.998 \cdot 299.8 \cdot 10^6 \text{ m} \approx 10.4 \text{ km}$$

which is bigger than the thickness of Earth's atmosphere explaining the detection. [15%]
(b) The muon must travel the distance $d = 10$ km in the time τ, which is the time dilated mean life τ_0 of the muon in its rest frame. It follows that

$$d \leqslant v\tau = v\gamma(v)\tau_0.$$

The velocity of the muon is given by $v = pc^2/E$, and thus, we have

$$\gamma(v) = \frac{1}{\sqrt{1 - (v/c)^2}} = \frac{E}{\sqrt{E^2 - p^2c^2}} = \frac{E}{mc^2}.$$

Inserting this into the above inequality, we obtain

$$d \leq \frac{pc}{mc^2} c\tau_0.$$

Inserting the numerical values, we find that $pc \gtrsim 15mc^2 \simeq 1.6$GeV, and thus, it holds that $E \simeq pc \gtrsim 1.6$GeV. [25%]
(c) A time interval for a muon and a time interval for an observer in the lab frame are related through the time dilation formula

$$\gamma(v) d\tau = dt,$$

where $d\tau$ is the time interval for the muon, dt is the time interval for the observer n the lab frame, and v is the muon velocity. The muon velocity is constant (since he total energy is constant) and given by

$$v = \frac{p}{E} = \frac{\sqrt{E^2 - m^2}}{E} = \sqrt{1 - \frac{m^2}{E^2}},$$

where p is the muon momentum, E is the muon energy, and m is the muon mass.

[15%]

It follows that

$$\gamma(v) = \frac{E}{m},$$

and thus, we obtain

$$t = \frac{E}{m}\tau = \frac{1\text{GeV}}{106\text{MeV}}\tau \simeq 9.5\tau.$$

The average lifetime of the muon in the lab frame is therefore 21μs. Since $\gamma(v)$ is known we determine $v/c \approx 0.995$, so the length travelled by the muon in the lab frame in the average lifetime is given by

$$\ell = vt \approx 0.995 \cdot ct \approx 0.995 \cdot 3 \cdot 10^8 \text{ m/s} \cdot 21 \cdot 10^{-6} \text{ s} = 6270 \text{ m}.$$

[15%]

The circumference of the circular accelerator is given by

$$L = 2\pi r \simeq 300 \text{ m}.$$

Thus, the average number of turns taken by a muon is given by

$$N = \frac{\ell}{L} \simeq 21.$$

[15%]

How cold is too cold?

Since the latent heat of ice is given, the skating process has to be modelled as a first order phase transition. This implies that (phase 1 is water, phase 2 is ice in what follows)

(2p)
$$\mu_1 = \mu_2 \to d\mu_1 = d\mu_2. \tag{1}$$

The chemical potential is also the reduced Gibbs potential, so

(1p)
$$-S_1 dT_1 + V_1 dp_1 = -S_2 dT_2 + V_2 dp_2. \tag{2}$$

The two phases are in thermal, chemical and mechanical equilibrium (by definition of a first order phase transition). This is expressed mathematically as

(1p)
$$p_1 = p_2 = p, \qquad T_1 = T_2 = T, \tag{3}$$

together with relation (1). Inserting (3) into (2) we obtain the Clausius-Clapeyron equation on the phase line

(1p)
$$\left(\frac{dp}{dT}\right)_{\text{phase line}} = \frac{S_2 - S_1}{V_2 - V_1}. \tag{4}$$

Moreover, at temperatures higher than those of phase transitions μ_1 ¡ μ_2 and at temperatures lower than those of phase transition μ_1 ¿ μ_2. In other words, there's no ice above the freezing point and no water below the freezing point. This implies that

(1p)
$$\left(\frac{\partial \mu_1}{\partial T}\right) < \left(\frac{\partial \mu_2}{\partial T}\right), \tag{5}$$

i.e., for any temperature, $S_1 > S_2$.

The lowest temperature permitted for enjoyable skating is the temperature at which the pressure on the coexistence line is equal to the pressure exerted by the skater on ice. For a skater of normal weight and a reasonable skating blade of sides 30 cm (to simplify calculations) by 1 mm, the pressure exerted by the skater is $p' \simeq 13$ atm (two blades per normal human). The specific volume of ice is larger than the specific volume of water, Using the pressure and temperature at the freezing point of water we obtain

(3p)
$$\frac{p' - p_0}{T_{\text{min}} - T_0} = -\frac{h}{T_{\text{min}} \Delta V}, \tag{6}$$

which can be inverted to obtain

(1p) $$T_{\min} = \frac{T_0}{1 + \frac{(p' - p_0)\Delta V}{h}} \simeq -0.09^o \text{ C} \qquad (7)$$

Solution
Neutrons in a box

a)

- the average time $<t>=<D>/v=<D>/\sqrt{2*E/m}$ (1p)
- the average distance travelled by a neutron before tunneling through a wall is

$< D >= d * T + 3 * d * R * T + 5 * d * R^2 * T + \cdots = d * (1 + 3 * R + 5 * R^2 + \cdots) * T$ (1p)

- $< D >= d * (R + 1) * T/(1 - R)^2 = d * (R + 1)/T$ (0.5p)
- $< t >= d * (R + 1)/ (T * \sqrt{2 * E/m})$ (0.5p)

b)

- the number of neutrinos that tunnel through the walls in a unit of time is equal to $N/< t >$ (1p)
- equilibrium is reached when $P = N/< t >$, thus $N = P *< t >= P * d * (R + 1)/(T * \sqrt{2 * E/m})$ (1p)

c)

- write the Schrodinger equation for the three sectors of the potential barrier (0.5p)
- solve all the three Schrodinger equations (1p)
- impose the continuity of the wave-function and of its first derivative (0.5p)
- solve the system of equation produced by the continuity conditions (1p)

d)

- calculate the currents of probability (1p)
- calculate the transmission and reflection coefficient (1p)

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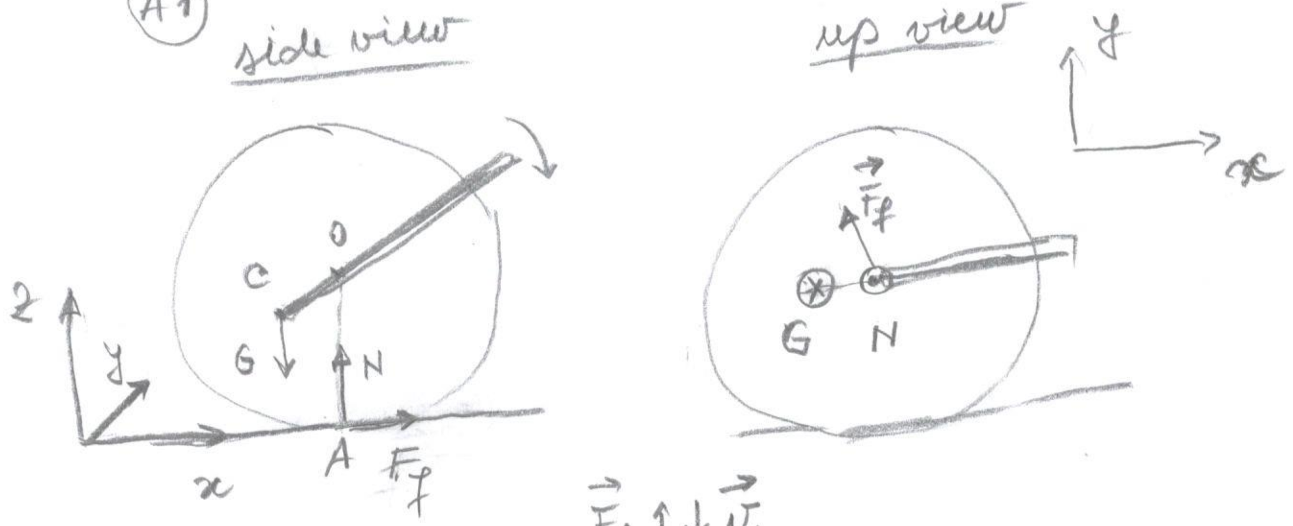
A1



$$\vec F_f \uparrow\downarrow \vec v_A$$

$$\vec F = m\vec a_C = (N - mg)\,\hat z + \vec F_f =$$

$$= (N - mg)\,\hat z - \frac{\mu_0 N}{|\vec v_A|}\,\vec v_A$$

No, C has a complex motion, there are external forces which make $\vec a_C \neq \vec 0$. The principal and central axes, that is, the Euler's system 123, because

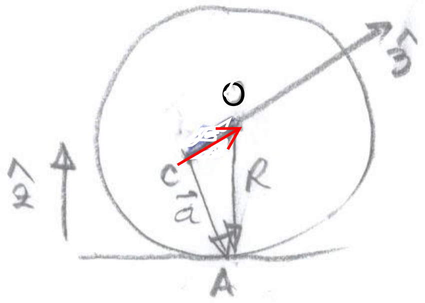
$$\hat I_C = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

and $\hat I_C \vec\varepsilon + \vec\omega \times \hat I_C \vec\omega = \vec{CA} \times (\vec N + \vec F_f)$ becomes simpler.

A2

$$\vec{\tau}_{ext} = \vec{CA} \times (N\hat{z} + \vec{F}_f) = \vec{a} \times (N\hat{z} + \vec{F}_f)$$

$$\vec{a} = \vec{CA} = \vec{CO} + \vec{OA} = \alpha R\hat{3} - R\hat{z}$$



$$\begin{pmatrix} \hat{1} \\ \hat{2} \\ \hat{3} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} =$$

x y

z

$$= \begin{pmatrix} \cos\theta\,\hat{x} - \sin\theta\,\hat{z} \\ \hat{y} \\ \sin\theta\,\hat{x} + \cos\theta\,\hat{z} \end{pmatrix}$$

$$\vec{\tau}_{ext} = (\alpha R\hat{3} - R\hat{z}) \times (N\hat{z} + F_{fx}\hat{x} + F_{fy}\hat{y}) =$$

$$= \alpha R\left[N\,\hat{3}\times\hat{z} + \alpha R\,\hat{3}\times(F_{fx}\hat{x} + F_{fy}\hat{y})\right]$$

$$- R F_{fx}\hat{y} + R F_{fy}\hat{x} =$$

$$= \alpha R\left[-N\sin\theta\,\hat{y} + F_{fx}\cos\theta\,\hat{y} + F_{fy}\sin\theta\,\hat{z} - F_{fy}\cos\theta\,\hat{x}\right]$$

$$- R F_{fx}\hat{y} - R F_{fy}\hat{x}$$

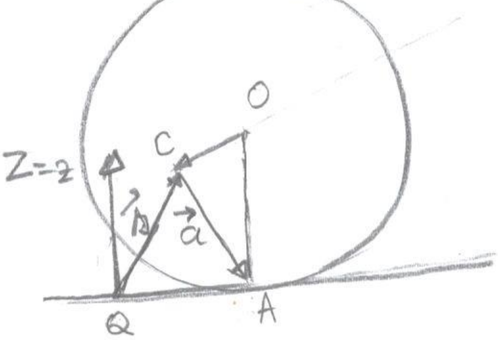
$$= - R(\alpha F_{fy}\cos\theta - F_{fy})\hat{x} + R(-\alpha N\sin\theta + \alpha F_{fx}\cos\theta$$

$$- F_{fx})\hat{y} + \alpha R F_{fy}\sin\theta\,\hat{z} =$$

$$= R F_{fy}(1 - \alpha\cos\theta)\hat{x} + R\left[\alpha(F_{fx}\cos\theta - N\sin\theta) - F_{fx}\right]\hat{y}$$

$$+ \alpha R F_{fy}\sin\theta\,\hat{z}$$

A3



$\vec{s} = \vec{QC}$; $\vec{a} = \vec{CA}$

$\vec{s} + \vec{a} = \vec{QA}$

La contact $QA \perp OA \Leftrightarrow$

$(\vec{s} + \vec{a}) \cdot \hat{z} = 0$

Chasle: $\vec{v}_A = \vec{v}_O + \vec{\omega} \times \vec{OA}$

$\vec{v}_O = \vec{v}_C + \vec{\omega} \times \vec{CO}$

$\Rightarrow \vec{v}_A = \vec{v}_C + \vec{\omega} \times (\vec{OA} + \vec{CO}) = \vec{v}_C + \vec{\omega} \times \vec{a}$

$\vec{v}_C = \dot{\vec{s}}$ $\Rightarrow$

$\Rightarrow \vec{v}_A = \dot{\vec{s}} + \vec{\omega} \times \vec{a}$

$\vec{a} = \vec{OA} + \vec{CO} = -R\hat{z} + \alpha R \hat{3}$

$\Rightarrow \vec{v}_A \cdot \hat{z} = \left[\dot{\vec{s}} + \vec{\omega} \times (-R\hat{z} + \alpha R \hat{3})\right] \cdot \hat{z}$

$= (\dot{\vec{s}} + \vec{\omega} \times \alpha R \hat{3}) \cdot \hat{z}$ $\Rightarrow$

$\dot{\vec{a}} = \alpha R \dot{\hat{3}} = \alpha R \vec{\omega} \times \hat{3}$

$\Rightarrow \vec{v}_A \cdot \hat{z} = (\dot{\vec{s}} + \dot{\vec{a}}) \cdot \hat{z} = \frac{d}{dt}\left[\underbrace{(\vec{s} + \vec{a}) \cdot \hat{z}}_{0}\right] = 0$

$\Leftrightarrow \vec{v}_A = \vec{v}_{Ax} + \vec{v}_{Ay}$

$= v_{Ax}\hat{x} + v_{Ay}\hat{y}$

A4 With Euler's angles: $\vec{\omega} = \dot{\phi}\hat{z} + \dot{\theta}\hat{y} + \dot{\psi}\hat{3}$

$\hat{3} = \sin\theta\,\hat{x} + \cos\theta\,\hat{z}$

$\Rightarrow \vec{\omega} = \dot{\psi}\sin\theta\,\hat{x} + \dot{\theta}\hat{y} + (\dot{\phi} + \dot{\psi}\cos\theta)\hat{z}$

From A2:

$$\begin{cases} \hat{1} = \cos\theta\,\hat{x} - \sin\theta\,\hat{z} \\ \hat{2} = \hat{y} \\ \hat{3} = \sin\theta\,\hat{x} + \cos\theta\,\hat{z} \end{cases} \Rightarrow \begin{cases} \hat{x} = \cos\theta\,\hat{1} + \sin\theta\,\hat{3} \\ \hat{y} = \hat{2} \\ \hat{z} = -\sin\theta\,\hat{1} + \cos\theta\,\hat{3} \end{cases}$$

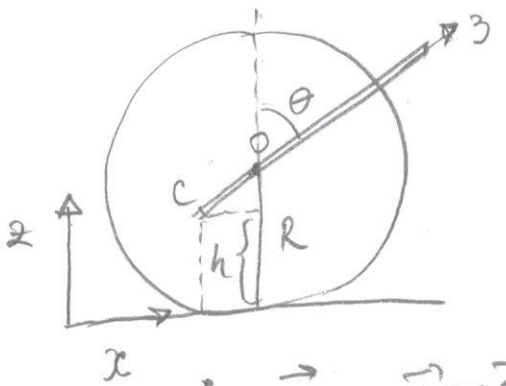
$$\Rightarrow \vec{w} = \dot\psi \sin\theta(\cos\theta\,\hat{1} + \sin\theta\,\hat{3})$$
$$+ \dot\theta\,\hat{2} + (\dot\phi + \dot\psi\cos\theta)(-\sin\theta\,\hat{1} + \cos\theta\,\hat{3})$$
$$= \hat{1}(\dot\psi\sin\theta\cos\theta - \dot\phi\sin\theta - \dot\psi\cos\theta\sin\theta)$$
$$+ \hat{2}\,\dot\theta +$$
$$\hat{3}(\dot\psi\sin^2\theta + \dot\phi\cos\theta + \dot\psi\cos^2\theta) =$$
$$\hat{3}(\dot\psi + \dot\phi\cos\theta) =$$
$$= -\hat{1}\,\dot\phi\sin\theta + \hat{2}\,\dot\theta + \hat{3}(\dot\psi + \dot\phi\cos\theta)$$

(A 5) For kinetic energy one uses König theorem:

$$K = K_T + K_R = \frac{m}{2}\dot{\vec{s}}^{\,2} + \frac{1}{2}\vec{w}\cdot \hat{I}_c\,\vec{w}$$

The gravitational potential energy:

$$U_G = mgh = mg(R - \alpha R\cos\theta) = mgR(1 - \alpha\cos\theta)$$



$\dot{\vec{s}} = \vec{v}_A - \vec{w}\times\vec{a}$ from A 3

$\vec{a} = \alpha R\hat{3} - R\hat{z}$ from A 2

$\vec{w} = \hat{z}\,\dot\phi + \hat{y}\,\dot\theta + 3\,\dot\psi$ from A 4

$$\Rightarrow \vec{s} = \vec{v}_A - \underbrace{(\hat{z}\dot{\phi} + \hat{y}\dot{\theta} + 3\dot{\psi})}_{\vec{w}} \times \underbrace{(\mathcal{L}R\hat{3} - R\hat{z})}_{\vec{a}}$$

$$= [v_{Ax} + \dot{\theta}R(1-\mathcal{L}\cos\theta)]\hat{x} + [v_{Ay} - R\sin\theta(\mathcal{L}\dot{\phi} + \dot{\psi})]\hat{y}$$

$$+ \dot{\theta}\mathcal{L}R\sin\theta\,\hat{z}$$

$$\vec{w}\cdot \hat{I}_c \vec{w} = (w_1, w_2, w_3)\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$= (w_1, w_2, w_3)\begin{pmatrix} I_1 w_1 \\ I_2 w_2 \\ I_3 w_3 \end{pmatrix} = I_1 w_1^2 + I_2 w_2^2 + I_3 w_3^2$$
$$= I_1(w_1^2 + w_2^2) + I_3 w_3^2$$

$$\Rightarrow E_T = K + U_G =$$

$$= \frac{1}{2}\left[I_1(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\right]$$

$$+ \frac{m}{2}\Big\{[v_x + \dot{\theta}R(1-\mathcal{L}\cos\theta)]^2 + [v_y - R\sin\theta(\mathcal{L}\dot{\phi} + \dot{\psi})]^2$$

$$+ \dot{\theta}^2\mathcal{L}^2R^2\sin^2\theta\Big\} + mgR(1-\mathcal{L}\cos\theta).$$

A6 $\dot{\vec{L}}\cdot\hat{z} = \vec{\tau}\cdot\hat{z}$

With $\vec{\tau} = \vec{\tau}_{ext}$ from A2 $\Rightarrow$

$$\dot{\vec{L}}\cdot\hat{z} = \mathcal{L}RF_{fy}\sin\theta$$

A7 $\vec{L} = \hat{I}\vec{w} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} I_1 w_1 \\ I_2 w_2 \\ I_3 w_3 \end{pmatrix} =$

from A4 p. 4

$$= \begin{pmatrix} -I_1 \dot\phi \sin\theta \\ I_1 \dot\theta \\ I_3 (\dot\psi + \dot\phi \cos\theta) \end{pmatrix}$$

$$\Rightarrow \vec{L} \times \hat{3} = -I_1 \dot\phi \sin\theta \, \hat{1} \times \hat{3} + I_1 \dot\theta \, \hat{2} \times \hat{3} =$$

$$= + I_1 \dot\phi \sin\theta \, \hat{2} + I_1 \dot\theta \, \hat{1}$$

$$= I_1 (\dot\phi \sin\theta \, \hat{2} + \dot\theta \, \hat{1})$$

$$= I_1 \vec{w} \times \hat{3}$$

$k = I_1.$