PLANCK'S preliminaries, March 2024, ROMANIA

Comisia de concurs

Prof. dr. Ionel Lazanu; *Prof. dr. Lucian Ion*; *Prof. dr. Mihai Dima*; *Conf. dr. Petrică Cristea*; *Conf. dr. Tiberius Cheche*; *Conf. dr. Radu Slobodeanu*; *Lect. dr. Marian Băzăvan* ; *Lect. dr. Mihai Marciu* ; *Lect. dr. Stefan Ghinescu*; *Drd. Vasile Sevestrean*

Nuclear and Particle Physics

i) The strong nuclear force is transmitted between a proton and neutron by the creation and exchange of a pion. Taking the range of the strong nuclear force to be about 1 fermi (10^{-15} m), Yukawa calculated the approximate mass of the pion carrying the force, assuming it moves at nearly the speed of light. Please estimate the mass of pion, reproducing the original, but simplified calculation of the Yukawa.

2 p

ii) Please explain if this particle is observable or not.

1 p

iii) What is the minimum kinetic energy necessary as the process $p + n \to p + n + \pi^0$ to be produced. The threshold energy corresponds to minimum energy for direct observable of the neutral pion. $m_p = m_n \cong 940\ MeV/c^2$; $m_\pi \cong 135\ MeV/c^2$

2 p

iv) The neutron is an unstable particle. Please draw the decay process at quark level, putting in evidence the correct propagator of the dominant force.

Specify the time flow axis and show the conservation of electric charge in each vertex. How is it possible that for a neutron and a proton, particles with masses of about 0.939 GeV/c^2, respectively 0.938 GeV/c^2, the propagator W has a mass of about 83 GeV/c^2.

2 p

v) What was the dominant interaction (force) in this case. Calculate the radius of action of this interaction in the considered disintegration.

2 p

Oficiu: 1 p

Thermodynamics

The state equation of a thermodynamic system is:

$$p = \frac{AT^2}{V} \tag{1}$$

in which p, V and T represent pressure, volume and temperature, whereas A is a constant. The expression of the internal energy of the system is provided by the relation:

$$U = BT^n ln\left(\frac{V}{V_0}\right) + f\left(T\right) \tag{2}$$

in which B, n and V_0 are constants, whereas $f\left(T\right)$ is a function which depends only on temperature. Find the values of B and n.

Electron motion under the weak influence of a circular current loop

An electron moves within a close region of the rotational x-axis of a circular current loop (radius R, intensity I). When far away from the loop center, the electron moves at a velocity $\mathbf{v}$ parallel to the x-axis. The coil influence on electron velocity $\mathbf{v}$ is negligible, except for a small region close to the center of the coil. The initial velocity is not considerably altered during electron motion, but is not a constant.

Compute the angle that the electron will have rotated around x-axis with respect to its initial (far away) position, long after passing through the coil.

Thin reflective and anti-reflective layers

A. A film of soapy water ($n = 4/3$) is illuminated with monochromatic optical radiation such at an angle of incidence is $30°$. Maximum interference is observed when the light has a wavelength of 640 nm, and minimum interference occurs when the wavelength is 400 nm.

Derive the mathematical expression of the minimum film thickness and calculate its numerical value.

2 p

B. Consider the normal incidence of light on the separation surfaces between the optical media. A dielectric layer with thickness t and refractive index n_1 is deposited on the surface of a glass lens ($n = 1.5$). The light originates from air ($n_0 = 1$), traverses the layer, and then the lens.

1. Derive the expression for the energy reflection coefficient of the input face-layer assembly of the lens

2 p

2. Using a light beam of wavelength $\lambda_0 = 500$ nm (in vacuum), deduce the refractive index and the minimum thickness of the layer for it to be antireflective.

1 p

C. Let us consider a structure comprising multiple (double) layers of transparent dielectric, with alternating refractive indices n_1, n_2 ($n_1 > n_2$), as in the figure. The thicknesses corresponding to the layers are l_1, l_2.

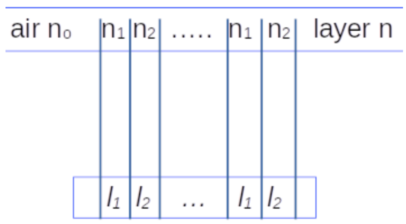

Figure 1: A structure comprising multiple (double) layers of transparent dielectric

1. Derive the expression for the energy reflection coefficient for such a structure, consisting of N such double layers, where the thicknesses correspond to a phase difference $\pi/2$.

2 p

2. If such a double layer is deposited on the surface of the lens at point **B**, deduce the condition under which this double layer does not allow the reflection of light. Compare this result with the one from point **B**, what provide your observations.

1 p

3. Calculate the reflection coefficient for a single double layer for the case $n_1 = 2.4$ (TiO_2 titanium dioxide) and $n_2 = 1.38$ (MgF_2 magnesium fluoride). Assume $n = n_0 = 1$.

1 p

4. Determine the number of $TiO_2 - MgF_2$ double layers required to achieve a reflection coefficient greater than 95%. Assume $n = n_0 = 1$.

1 p

Conductivity in disordered electronic systems

Introduction

In some conditions, energy bands exists in disordered materials, including extended electronic states in the middle of the bands and localized states in the band tails (Fig. 1).

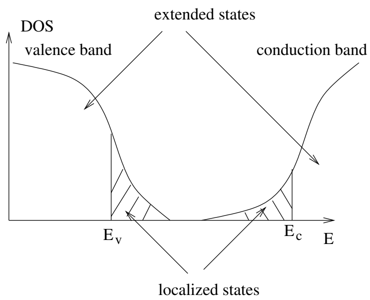

Figure 1: Density of states in a disordered material: electronic states in the band tails are localized, states in the middle of the bands are extended.

In the following, you are asked to study various conduction mechanisms, corresponding to extended or localized states.

Electrical conductivity at T = 0 K

1. Because of disorder, the mean free path l of charge carriers is finite. Consider first the contribution of the electrons occupying extended states ($k_F l \gg 1$), with energy E_α; those states can be expanded in terms of a discrete plane-wave basis set:

$$|\alpha\rangle = \sum_{\vec{k}} a_{\vec{k}}^{(\alpha)} |\vec{k}\rangle. \tag{1}$$

The real part of the electrical conductivity can be calculated by using Kubo-Greenwood formula:

$$\mathrm{Re}\,\sigma(\omega) = \frac{e^2\pi}{\omega\, m^2 V} \sum_{\alpha\neq\beta} |p|^2_{\alpha\beta}(f_\beta^{(0)} - f_\alpha^{(0)})\delta(E_\alpha - E_\beta - \hbar\omega), \tag{2}$$

where Re z is the real part of the complex quantity z, V is the volume of the system, $p_{\alpha\beta} = \langle\alpha|p|\beta\rangle$, and $f^{(0)}(E)$ is Fermi-Dirac distribution function. Show that at $T = 0K$ ec. (2) may be recast as:

$$\mathrm{Re}\,\sigma(\omega) = \frac{2\pi\, e^2 V\hbar}{m^2} \int_{E_F-\hbar\omega}^{E_F} \frac{|p_{\alpha\beta}|^2 n(E) n(E+\hbar\omega)}{\hbar\omega} dE, \tag{3}$$

where $n(E)$ is the density of 1-particle states (DOS) per spin. Supposing that DOS of the disordered system is not significantly modified as compared to that in the crystalline (ordered) system and that:

$$|p_{\alpha\beta}|^2 = \frac{2\hbar^2\pi\, l}{3V} \frac{1}{1+(k_\alpha - k_\beta)^2 l^2}, \tag{4}$$

determine $\mathrm{Re}\,\sigma(\omega)$ in the limit $\hbar\omega \ll E_F$. Show that the classical result (Drude formula) is recovered. Give a physical explanation for that result.

2. The mean free path in a conductor cannot be less than de Broglie's wavelength of electrons (criterion of Ioffe and Regel). Show that this criterion defines a minimum value of $\mathrm{Re}\,\sigma(0)$, *minimum metallic conductivity*, introduced first by Mott. Determine the minimum metallic conductivity and show that it is inversely proportional to the minimum mean free path.

3. Consider now the contribution of localized states. Consider two sites, a and b, separated by the distance R, where localized states $|a\rangle$ and $|b\rangle$ (corresponding to the same energy) are centered. If there is a superposition of the wave functions associated to those two states, the degeneracy is removed, and the eigenstates associated to the system of the coupled centers read:

$$\begin{aligned} |\psi_1\rangle &= \tfrac{1}{\sqrt{2}}\left(|a\rangle + |b\rangle\right) \\ |\psi_2\rangle &= \tfrac{1}{\sqrt{2}}\left(|a\rangle - |b\rangle\right) \end{aligned} \tag{5}$$

4. Show qualitatively that the energies associated to $|\psi_1\rangle$ and $|\psi_2\rangle$ are separated by the quantity (resonance energy)

$$2I = 2I_0 \exp(-KR), \tag{6}$$

K^{-1} being the localisation radius of localized states (suppose they are s-type orbitals).

5. Starting from eq. (3), show that the conductivity can be expressed as a function of the matrix elements of the position operator $\vec{r}$. (*Hint:* use the dynamical equation of $\vec{r}$ in the Heisenberg picture of the quantum mechanics).

6. In the limit $\hbar\omega \ll E_F$, in a disordered system, physical meaning has the average of the conductivity over all possible configurations of the sites $\vec{r_a}$ and $\vec{r_b}$, supposed uniformly distributed. Introducing $\vec{R} = \vec{r_a} - \vec{r_b}$, determine the expression of the average $\langle\sigma(\omega)\rangle$ as a function of $\vec{r_a}$, $\vec{R}$ and of the matrix elements $\vec{r}_{\alpha\beta} = \langle\alpha|\vec{r}|\beta\rangle$.

7. Determine $\vec{r}_{\alpha\beta}$ for small R ($KR \leq 1$), supposing s-type localized orbitals. Show that:

$$|\vec{r}_{\alpha\beta}| = \frac{R}{2}. \tag{7}$$

8. Show qualitatively that at large distances ($KR \gg 1$):

$$|\vec{r}_{\alpha\beta}| \propto R\, e^{-KR}. \tag{8}$$

9. In this regime, electrical conduction occurs through electron hopping between localized states separated by R. Determine the minimum hopping distance R_ω, knowing that the corresponding resonance energy is less than $\hbar\omega$.
Show that in this case the essential contribution to the average of the conductivity comes from the spherical shell of radii R_ω and $R_\omega + \frac{1}{K}$. As a consequence, show that:

$$\langle\sigma(\omega)\rangle \propto \omega^2(\log\omega)^4. \tag{9}$$

Relativistic particles

The average lifetime of the muon in its proper/rest frame is $2.2 \times 10^{-6} s$, its rest mass is 106 MeV.

(a) Assuming that muons travel at 99.8% of the speed of light, show that cosmic radiation muons can indeed be detected on the surface of the Earth. Take $d = 10$km as thickness of Earth's atmosphere and $299.8 \cdot 10^6$m/s the speed of light. The direction of the muon is vertical with respect to the ground.

(b) What is the smallest energy required for muons to hit the ground before they decay, assuming that they are produced at an altitude of 10 km above ground? (here you drop the velocity assumption at point (a)!). The direction of the muon is vertical again.

(c) A circular accelerator has a radius of 50m. How many turns can a muon take on average in this ring before it decays if its energy is kept constant at 1 GeV? Here, take $c = 3 \cdot 10^8 m/s$.

How cold is too cold?

In an ice skating rink, skating becomes unpleasant (i.e. falling frequently) if the temperature is too cold so that the ice becomes too hard. Estimate the lowest temperature of the ice on a skating rink for which ice skating for a person of normal weight would be possible and enjoyable (the latent heat of ice is 334 J/g and water expands by 0.091 cm^3/g when it freezes). *Hint*: the freezing point of water occurs at $p_0 \simeq 1$ atm and $T_0 \simeq 273.16$ K.

Neutrons in a box

We have a point-like non-relativistic neutron source in a one-dimensional system. At distance d from the source, there are walls on both sides of the source. The source emits isotropically P neutrons per second of energy E. The wall can be modelled as a one-dimensional potential barrier of height $V0 = 2 * E$ and width a ($d >> a$). The neutron reflection and transmission probabilities at each collision with the walls are R and T, respectively. We want to find the number of neutrons between the two walls at equilibrium (a long time after the source has been turned on). The Planck constant – h and the mass of the neutron - m are known.

a) Compute the average time a neutrons spends in between the walls. (3p)

b) Calculate N - the number of neutrons between the walls at equilibrium. (2p)

c) Calculate the energy eigenfunctions (wave-functions) of the neutrons in the barrier and its vicinity without the normalization condition. (3p)

d) Calculate the reflection and the transition probability. (2p)

$$\sum_{i=0}^{\infty} i\ X^i = \frac{X}{(1-X)^2} \quad \text{when} \quad |X| < 1 \tag{1}$$

Tippe top[1]

Fig.1 Tippe top photo

A Tippe top is a special kind of top that can spontaneously invert once it has been set spinning (see the side Fig.1). *There is no fixed point of the top and the friction force between the top and the surface it is moving on is the driving force.*

One can model a Tippe top as a sphere of radius R that is truncated, with a stem added. It has rotational symmetry about the symmetry axis (SA) passing through the stem, which is at angle θ from the vertical. As shown in Fig. 2(a), its centre of mass C is offset from its geometric centre O by $\|CO\|=\alpha R$ along SA. The Tippe top makes contact with the planar surface (floor) it rests on at point A. If the Tippe top spins fast enough initially about the (approximately) vertical SA, then it will tip so that the stem points increasingly downwards, until it starts to spin on in its stem (the sphere is above the stem which is in contact with the floor), and eventually stops to the equilibrium position shown in Fig. 1.

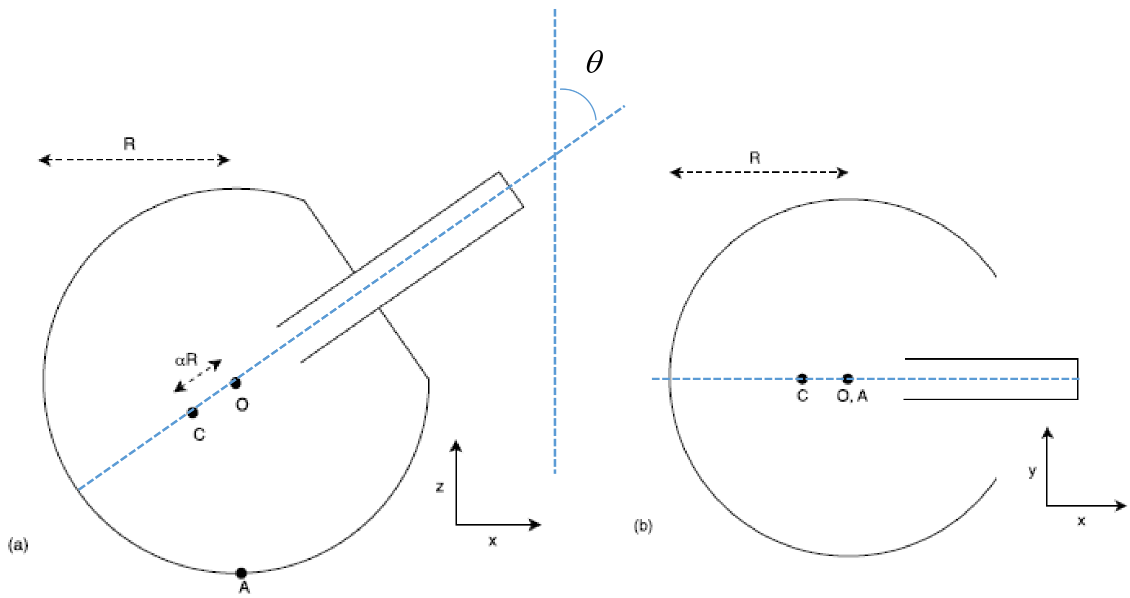

Fig 2. Views of the Tippe top (a) from the side and (b) from above

Let xyz be the *rotating* reference frame defined such that the unit vector $\vec{\mathbf{z}}$ of z axis is stationary and upwards, and the top's symmetry axis is within the xz-plane. Two views of the Tippe top are shown in Fig. 2: from the side, and from above. As shown in Fig. 2(b), the top's SA is aligned with the x-axis when viewed from above.

[1] *The problem is adapted from Asian Physics Olympiad 2019*

Fig. 3 shows the top's motion at several phases after it is started spinning:
(a) **phase I:** immediately after it is initially set spinning, with $\theta \sim 0$
(b) **phase II:** soon after, having tipped to angle $0 < \theta < \pi$ 2
(c) **phase III:** when the stem first touches the floor, with $\theta > \pi$ 2
(d) **phase IV:** after inversion, when the top is spinning on its stem, with $\theta \sim \pi$
(e) **phase V:** in its final state, at rest on its stem $\theta = \pi$.

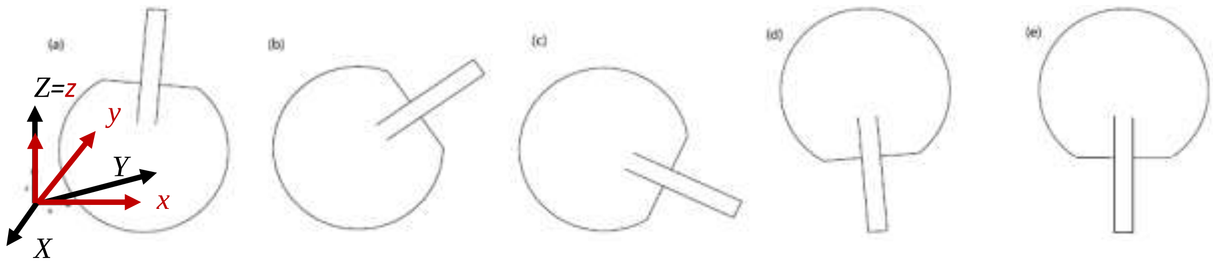

Fig 3. Phases **I** to **V** of the Tippe top's motion, shown in the xz-plane

Let XYZ be the fixed inertial frame, where the surface the top is on is wholly in the XY-plane. The above defined frame xyz is reached from XYZ via rotation around the Z axis by ϕ. The transformation from the XYZ frame to frame xyz is shown in Figure 4(a). In particular, $\vec{z} = \vec{Z}$.

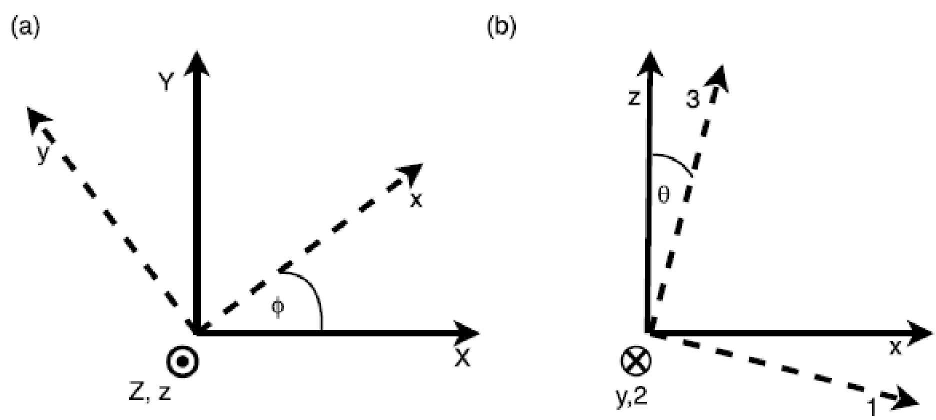

Fig. 4. Transformations between reference frames: (a) from XYZ to xyz, and (b) to 123 from xyz.

Any rotational motion in 3-dimensional space can be described by the three Euler angles (θ, ϕ, ψ). The transformations between the inertial frame XYZ, the intermediate frame xyz, and the top's frame 123 can be understood in terms of these Euler angles. In the XYZ frame they are defined as follows: θ is the angle of the top's symmetry axis from the vertical Z-axis, representing how far from vertical its stem is, while ϕ represents the top's angular position about the Z-axis, and is defined as the angle between the XZ-plane and the plane through points O, A, C (i.e. the vertical projection of the top's symmetry axis). The third Euler angle ψ describes the rotation of the top about its own symmetry axis, i.e. its 'spin', which has angular velocity $\dot{\psi}$ (generally for a physical quantity q, we denote $\dot{q} \equiv dq/dt$ as the time derivative of q). The reference frame of the spinning top rigidly attached to the top is defined as a new rotating frame 123, which is reached by rotating

xyz by θ around $\vec{y}$: 'tilting' the $\vec{z}$ axis down by θ to meet the top's symmetry axis $\vec{3}$. The transformation from the xyz frame to the 123 frame is shown in Figure 4(b). In particular, $\vec{2} = \vec{y}$.

Assume that the top remains in contact with the floor at point A, until such time as the stem contacts the floor. It is in motion at point A with velocity $\vec{v}_A$ relative to the floor. The frictional coefficient μ_k between the top and floor is kinetic, with $\left|\overrightarrow{F_f}\right| = \mu_k \left|\vec{N}\right| = \mu_k N$, where $\overrightarrow{F_f} = F_{f,x}\vec{x} + F_{f,y}\vec{y}$ is the frictional force, and N is the magnitude of the normal force. Assume that the top is initially set spinning only, i.e. there is no translational impulse given to the top.

Let the mass of the Tippe top be m. Its moments of inertia of the principal central axes are: I_3 about the axis of symmetry, and $I_1 = I_2$ about the mutually perpendicular principal axes. Let $\vec{s}$ be the position vector of the centre of mass, and $\vec{a} = \overrightarrow{CA}$ be the vector from the centre of mass to the point of contact.

Next, one investigates the physics of the Tippe top to set up the system of equations of motion.

A1. Find the total external force $\overrightarrow{F_{ext}}$ on the Tippe top. Draw a free body diagram of the top, projected onto each of the xz and xy planes. Indicate the direction of $\overrightarrow{v_A}$ in the space provided, on your diagram in the xy-plane. Does the center of mass C keep its position during the motion relative to XYX? **1pt**

A2. Find the total external torque $\overrightarrow{\tau_{ext}}$ on the Tippe top about the centre of mass. **1.5pt**

A3. Explain why the contact condition can be written as $\left(\vec{s} + \vec{a}\right)\cdot\vec{z} = 0$. By using this contact condition show that the velocity at A has no component in the z-direction, i.e. we can write $\overrightarrow{v_A} = v_{Ax}\vec{x} + v_{Ay}\vec{y}$. **0.5pt**

A4. Find the total angular velocity ω of the rotating top about its centre of mass C in terms of the time derivatives of the Euler angles: $\dot{\theta}, \dot{\phi}, \dot{\psi}$. Use Fig. 4 if this is helpful. Give your answer in both xyz and 123 frames. **2pt**

A5. Find the total energy of a spinning Tippe top, in terms of time derivatives of the Euler angles, v_{Ax}, v_{Ay}. **2pt**

A6. Find the rate of change of the angular momentum about the z-axis. **1pt**

A7. Show that the components of the angular momentum $\vec{L}$ and angular velocity $\vec{\omega}$ that are perpendicular to the $\vec{3}$ direction are proportional, i.e. $\vec{L}\times\vec{3} = k\left(\vec{\omega}\times\vec{3}\right)$, and find the proportionality constant k. **1pt**