PLANCKS
Physics League Across Numerous Countries for Kick-ass Students

PLANCKS 2014

PHYSICS LEAGUE ACROSS NUMEROUS COUNTRIES FOR KICKASS STUDENTS

EXERCISES

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PLANCKS 2014:

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Table of contents

Introduction

Dear contestants,

In front of you, you have the exercises of PLANCKS 2014. You are about to compete for being the best physics student team of the world! I hope you will enjoy the competition.
Before you start working at the exercises, a few remarks must be made.

- Teams can consist of 3 or 4 (under-)graduate students
- The language used in the competition is English.
- The contest consists of 10 exercises, each is worth **10 points**. Subdivisions of points are indicated in the exercises.
- **All exercises have to be handed in separately.**
- When a problem is unclear, a participant can ask, through the crew, for a clarification from the jury. The jury will respond to this request. If this response is relevant to all teams, the jury will provide the other teams this information.
- You are allowed to use a dictionary: English to your native language.
- You are allowed to use a simple calculator (non-graphical and not scientific).
- The use of hardware (including phones, tablets etc.) is not approved, with exceptions of simple watches and medical equipment.
- No books or other sources of information are to be consulted during the competition.
- The organisation has the right to disqualify teams for misbehaviour or breaking the rules.
- In situations to which no rule applies, the organisation decides.

I wish you all the very best at the competition. May the best physics students team win PLANCKS 2014!

Felix Nolet
PLANCKS organisation
Commissioner Jury & Exercises

1 - Graphene

Carlo Beenakker, Leiden University

Graphene is a mono-atomic layer of carbon atoms, arranged in a honeycomb lattice. Conduction electrons move in the $(x, y)-$plane of the layer, with velocity $v = 10^6\,\mathrm{m/s}$ that is independent of their energy E.

[1] *2 point* Explain why the conduction electrons in graphene are called "massless particles".

Because the honeycomb lattice has two atoms in the unit cell, the wave function of the conduction electrons has two components $\Psi_1(x, y)$ and $\Psi_2(x, y)$. The two components satisfy a pair of coupled quantum mechanical wave equations,

$$\begin{cases} U(x,y)\Psi_1(x,y) + (vp_x - ivp_y)\Psi_2(x,y) = E\Psi_1(x,y), \\ U(x,y)\Psi_2(x,y) + (vp_x + ivp_y)\Psi_1(x,y) = E\Psi_2(x,y). \end{cases} \tag{1}$$

The electrical potential energy is indicated by $U(x, y)$ and $p_x = -i\hbar\partial/\partial x$, $p_y = -i\hbar\partial/\partial y$ are the two components of the momentum operator. For a uniform potential $U(x, y) \equiv U_0$ the solutions are proportional to the plane wave $\exp(ik_x x + ik_y y)$.

[2] *4 points* Derive, for this case of uniform potential U_0, the relation between the energy E and the wave vector components k_x, k_y. Make a plot of E as a function of $k \equiv \sqrt{k_x^2 + k_y^2}$. The singularity at $k = 0$ is called "conical point" or "Dirac point", and is a unique feature of graphene.

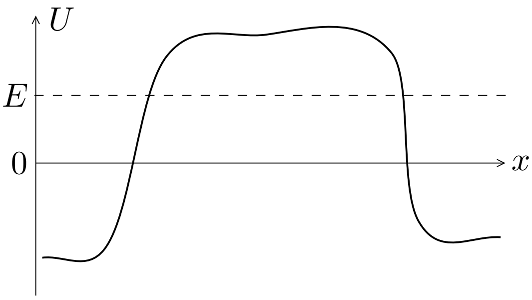

Figure 1: Plot of potential U(x)

We introduce a potential barrier $U(x)$ (see figure 1). An electron moves at energy E along the x-axis towards the barrier and is reflected by it with some probability R.

[3] *4 points* Verify this (y-independent) solution of the wave equation:

$$\begin{cases} \Psi_1(x) = C\exp\left(\frac{is}{\hbar v}\int_0^x [E - U(x')]dx'\right) \\ \Psi_2(x) = s\Psi_1(x) \end{cases} \tag{2}$$

with C an arbitrary constant and s equal to $+1$ or -1.

[4] *3 points* Show that the reflection probability $R = 0$ at any energy no matter how high the potential barrier.
This surprising result is known as the Klein paradox.

2 - Newton's Cradle

Jan van Ruitenbeek, Leiden University

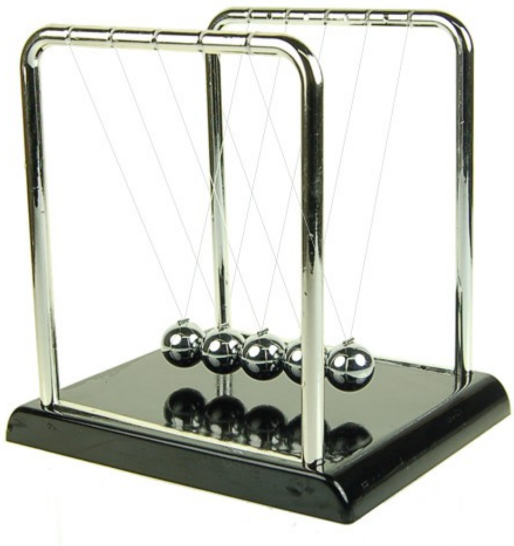

Figure 2: Newton's Cradle

Newton's cradle is a well-known gadget and physics demonstration. It is usually described as demonstrating the laws of conservation of energy and conservation of momentum.
For simplicity we take the motion to be one-dimensional and the collisions to be elastic.

[1] *5 points* We launch a single ball onto the other balls that are at rest, and consider the situation just after the collision. For any number N of balls (including the launched ball) in the cradle how many solutions do the laws of conservation of energy and momentum permit? For $N = 2$ and $N = 3$ describe the set of allowed solutions in N-dimensional velocity space.

[2] *5 points* When we perform the experiment for $N = 3$ we find that only one solution is realised. Which solution is this, and explain why.

3 - 2DEG at the AlGaAs-GaAs interface

Ingmar Swart, Utrecht University

A 2-dimensional electron gas (2DEG) can exist at the interface between semiconductors. One example where this naturally occurs is at the AlGaAs-GaAs interface. Due to the bending of the energy bands, the potential energy landscape has the shape as shown by the dashed line in figure 3. To first order, the area where the 2DEG forms, can be approximated by a triangular shaped potential well: $V(x < 0) = \infty$ and $V(x > 0) = Fx$, where F is a proportionality constant which has dimensions of force. It will be of the order of 10 $\frac{meV}{nm}$, or 1 pN. In this case, the problem can be solved analytically.

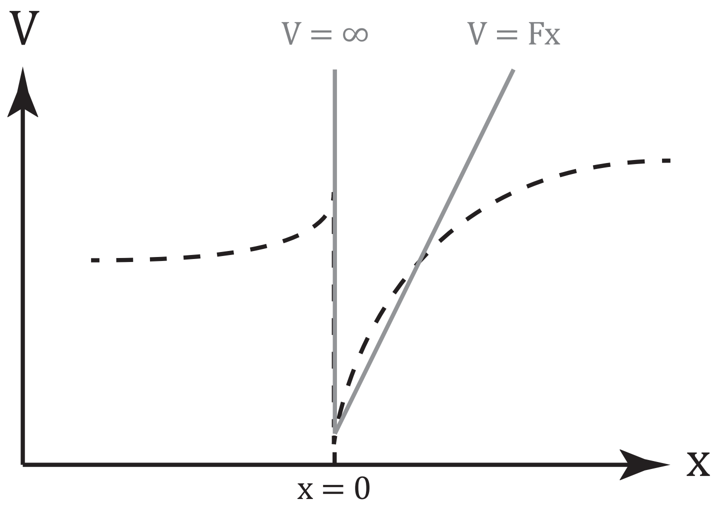

Figure 3: Schematic illustration of the potential energy landscape of a AlGaAs-GaAs interface (black dashed line). The area where the 2DEG is formed can be approximated by a triangular barrier (gray solid line).

For the wavefunction, use the ansatz where k is the wave vector.

$$\begin{cases} \psi_k(x) = xe^{-kx/2} & For\ x \geq 0 \\ \psi_k(x) = 0 & For\ x < 0 \end{cases} \tag{3}$$

[1] *10 points* Show that the expectation value of the ground state energy using this ansatz is given by:

$$E_g = 2.48 \left(\frac{\hbar^2}{2m^*}\right)^{1/3} F^{2/3} \tag{4}$$

with m^* the effective mass.
You may find the following integral helpful.

$$\int_0^\infty x^n e^{kx} dx = (-1)^n \frac{d^n}{dk^n} \int_0^\infty e^{-kx} dx = \frac{n!}{k^{n+1}} \tag{5}$$

4 - Exercises on Particle Physics

André Mischke, Utrecht University

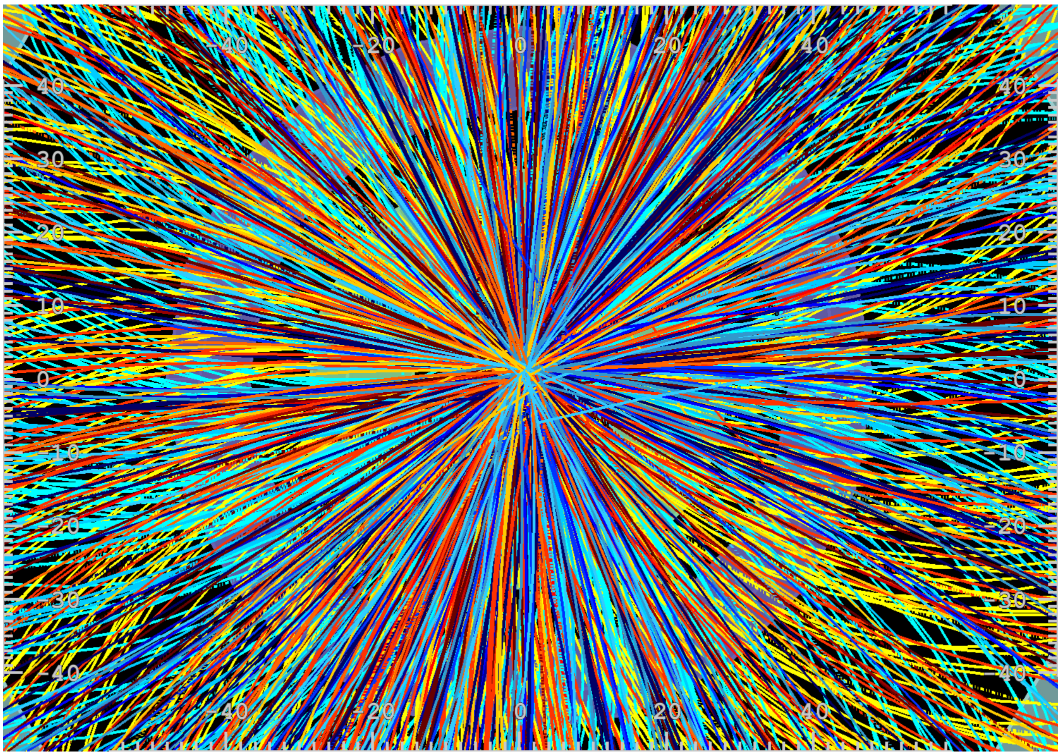

Figure 4: Tracks of particles from a collision of lead atomic nuclei, reconstructed by the ALICE experiment at the CERN Large Hadron Collider.

Charged particles in magnetic fields

[1] *2 points* A proton beam with a kinetic energy of 200 GeV passes through a 2 m long dipole magnet with a field strength of 2 T. Calculate the deflection angle Θ of the beam using the relation $2\sin\frac{\Theta}{2} = L/R$ and the momentum $p = 0.3 \cdot qBR$, where L is the length, R the radius, q the charge, and B the magnetic field.
The mass of the proton can be found in the table 1.
Note SI units!

[2] *2 points* In a high energy reaction a proton with a kinetic energy of 10 MeV is bended in a dipole magnet by $10°$ on a length $L = 2$ m. Calculate the necessary field using the relation from part **[1]**.

Particle identification

In particle collisions hundreds of new particles are produced (see figure 4). Particle identification (PID) is necessary to answer particular physics questions. In the following, pions (π), kaons (K) and protons (p) are identified using a Time-of-Flight (ToF) detector. The momentum p of the particles is measured by a tracking detector.

[3] *2 points* Find an expression for the mass m is a function of the momentum p, the flight length l and the flight time t. Given that the particles move on straight tracks.

[4] *2 points* The distance between the tracking detector and the ToF is l = 30 m. Identify the following particles using the equation obtained in **[3]** and the masses given in table 1 .

particle 1	p = 3.41 GeV/c	t = 103.71 ns
particle 2	p = 3.72 GeV/c	t = 100.86 ns
particle 3	p = 5.48 GeV/c	t = 100.39 ns
particle 4	p = 8.77 GeV/c	t = 100.013 ns

Reconstruction of short-lived particles

A liquid hydrogen target is bombarded with a $|\vec{p}|$ = 12 GeV/c proton beam. The momentum of the reaction products are measured in wire chambers inside a magnetic field.
In one event six charged particle tracks are seen. Two of them go back to the interaction vertex. They belong to positively charged particles. The other tracks come from two pairs of oppositely charged particles. Each of these pairs appears a few centimetres away from the interaction point. Evidently two electrical neutral, and hence unobservable, particles were created, which later both decayed into pairs of charged particles.
The measured momenta of the decay pairs were:

$|\vec{p}_+|$ = 0.68 GeV/c $|\vec{p}_-|$ = 0.27 GeV/c $\measuredangle(\vec{p}_+, \vec{p}_-) = 11^\circ$
$|\vec{p}_+|$ = 0.25 GeV/c $|\vec{p}_-|$ = 2.16 GeV/c $\measuredangle(\vec{p}_+, \vec{p}_-) = 16^\circ$

[5] *2 points* Use the method of *invariant mass* to determine the neutral particles.
Hint: Possible decay candidates are $\Lambda^0 \to$ p+π^- and $K_s^0 \to \pi^+ + \pi^-$. The mass of the proton p and pion $\pi^\pm$ are given in table 1.

name	Symbol	Mass (MeV/c^2)	Spin ($\hbar$)	Charge (e)	Antiparticle	Mean life-time (s)	Typical decay products*
Nucleon	p (proton) or N^+	938.3	1/2	+1	$\bar{p}$	$> 10^{32}$ y	
	n (neutron) or N^0	938.6	1/2	0	$\bar{n}$	930	$p + e^- + \bar{\nu}_e$
Lambda	Λ^0	1116	1/2	0	$\bar{\Lambda}^0$	2.5×10^{-10}	$p + \pi^-$
Sigma	Σ^+	1189	1/2	+1	$\bar{\Sigma}^-$	0.8×10^{-10}	$n + \pi^+$
	Σ^0	1192	1/2	0	$\bar{\Sigma}^0$	10^{-20}	$\Lambda^0 + \gamma$
	Σ^-	1197	1/2	-1	$\bar{\Sigma}^+$	1.7×10^{-10}	$n + \pi^-$
$Xi^\dagger$	Ξ^0	1315	1/2	0	$\bar{\Xi}^0$	3.0×10^{-10}	$\Lambda^0 + \pi^0$
	Ξ^-	1321	1/2	-1	$\bar{\Xi}^+$	1.7×10^{-10}	$\Lambda^0 + \pi^-$
Omega	Ω^-	1672	3/2	-1	Ω^+	1.3×10^{-10}	$\Xi^0 + \pi^-$
Charmed lambda	${\Lambda_c}^+$	2285	1/2	+1	$\bar{\Lambda c}$	1.8×10^{-13}	$p + K^- + \Lambda^+$
Pion	π^+	139.6	0	+1	π^-	2.6×10^{-8}	$\mu^+ + \nu_\mu$
	π^0	135	0	0	self	0.8×10^{-16}	$\gamma + \gamma$
	π^-	139	0	-1	π^+	2.6×10^{-8}	$\mu^- + \bar{\nu}_\mu$
Kaon	K^+	493.7	0	+1	K^-	1.24×10^{-8}	$\pi^+ + \pi^0$
	K^0	497.7	0	0	$\bar{K}^0$	0.88×10^{-10} and $5.2 \times 10^{-8\ddagger}$	$\pi^+ + \pi^-$ / $\pi^+ + e^- + \bar{\nu}_e$
Eta	η^0	549	0	0	self	2×10^{-19}	$\gamma + \gamma$

Table 1:
* Other decay modes also occur for most particles.
†The Ξ particle is somethimes called the cascade.
‡The K^0 has two distinct lifetimes, sometimes referred to as ${K^0}_{short}$ and ${K^0}_{long}$. All other particles have a unique lifetime.

5 - Laser cooling

Dries van Oosten, Utrecht University

An important technique in modern experimental physics is laser cooling and trapping. In this exercise, we will look at how laser cooling works.

We treat the atom as a two level system, with a groundstate $|g\rangle$ and an excited state $|e\rangle$. We write the energy difference between the ground- and excited state as $E_e - E_g = \hbar\omega_{eg}$ with ω_{eg} the transition frequency. We take the lifetime of the excited state to be $1/\gamma$. The laser beam has an intensity I and a frequency ω. We define the detuning of the laser with respect to the atomic transition frequency as $\delta = \omega_{eg} - \omega$. The intensity is often written in term of the saturation parameter s_0 and the saturation intensity I_s, which is a property of the atom, as $I = s_0 I_s$. Using this notation, the propability of the atom being in the excited state with the laser on-resonance (*i.e.* $\delta = 0$) can be written as

$$P_e = \frac{1}{2}\frac{s_0}{1+s_0} \tag{6}$$

In the case that the laser is off-resonance, we replace s_0 by $s = s_0/(1+4(\delta/\gamma)^2)$.

[1] *1 point* In the case that the laser is resonant, and in the limit that $s_0 \to \infty$, what is the rate by which the atom scatters photons?

[2] *1 point* In this case, what is the force exerted on the atom averages over many photon scattering events?

[3] *1 point* What is the force on the atom when $s_0 \ll 1$ and $\delta = 0$?

[4] *2 points* Now derive the force in the case that $\delta \neq 0$. Allow for the atom to have a velocity $\mathbf{v}$. Make a sketch of the force as a function of $\mathbf{v}$ for the case that $\delta < 0$ and the case that $\delta > 0$.

[5] *2 points* Now assume there is an identical laser beam counter propagating with the original laser beam. Derive the force. You may neglect the effects of interference between the two beams.
Again, make a sketch of the force as a function of $\mathbf{v}$ for the case that $\delta < 0$ and the case that $\delta > 0$.

[6] *2 points* Expand the expression you found in **[5]** for small $\mathbf{v}$, such that you have a force that is linear in $\mathbf{v}$. What type of force is this?

[7] *1 point* Now, we plug in some numbers for Rubidium-87. The transition frequency for Rubidium is $\omega = 2\pi \cdot 384$ THz and the linewidth $\gamma = 2\pi \cdot 6$ MHz. Calculate the force on the atom in the limit of exercise **[2]** and determine the resulting acceleration. Express the acceleration in terms of the earths gravitional acceleration g.

6 - No-cloning theorem

Lieven Vandersypen, Delft University of Technology

It is possible to clone sheep, but can you clone an unknown quantum state?

Imagine a friend gives you a particle in the state $|\psi\rangle$ without telling what state she is giving you. You want to make a faithful copy of this state, i.e. you want to have another particle with the exact same state. The other particle is initially in a state $|s\rangle$ that is independent of $|\psi\rangle$.

First assume that you have a quantum copy machine described by a unitary time evolution U, which acts in the following way on the two particles:

$$U(|\psi\rangle_1 |s\rangle_2) = |\psi\rangle_1 |\psi\rangle_2 \tag{7}$$

where the subscripts refer to particle 1 and 2 respectively. Similarly, if your friend gave you the state $|\phi\rangle$, the action of the copy machine would be

$$U(|\phi\rangle_1 |s\rangle_2) = |\phi\rangle_1 |\phi\rangle_2 \tag{8}$$

Note: you can answer questions 5 and 6 without knowing the answer to questions 1-4.

[1] *1 point* What is the inner product of $|\psi\rangle_1 |\psi\rangle_2$ and $|\phi\rangle_1 |\phi\rangle_2$?

[2] *1 point* What is the inner product of $U(|\psi\rangle_1 |s\rangle_2)$ and $U(|\phi\rangle_1 |s\rangle_2)$?

From the equalities above, it is clear that the inner products of questions 1 and 2 must be the same.

[3] *2 points* What constraints does this impose on $|\psi\rangle$ and $|\phi\rangle$?

[4] *2 points* What are the implications for cloning unknown quantum states?

[5] *3 points* We can also try to clone states using non-unitary processes, including measurements. The first idea that comes to mind is to measure the state of the particle we want to clone, and then to prepare multiple other particles in that same state. Considering spin-1/2 particles, either show that this works, or argue why it doesn't work.

[6] *1 point* Show that if quantum cloning were possible, it would be possible to communicate faster than light.

Hint: consider a so-called Bell pair, with two spin-1/2 particles in the state $(|\uparrow\rangle |\downarrow\rangle - |\downarrow\rangle |\uparrow\rangle)/\sqrt{2}$. Alice (on earth) possesses one of the particles, and Bob (on a planet light years away) the other.

7 - Connecting the dots

Henk Blöte, Leiden University

Some two-dimensional problems in statistical physics, such as a system of polymers, and the Ising, XY and Heisenberg models, can be formulated in terms of a sum on all configurations of non-intersecting loops in a plane. In the study of these loop models, the following problem occurs.

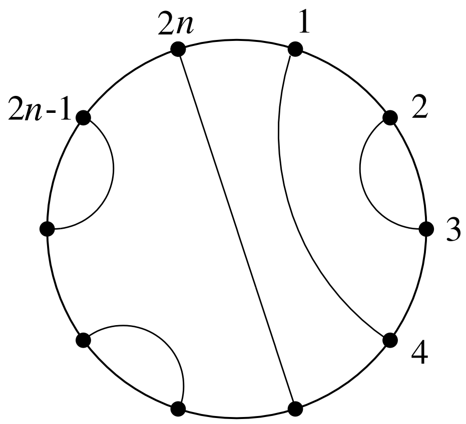

Figure 5: An example of non-intersecting loops in a plane.

Consider a circle, with $2n$ points on its circumference. These points are connected by n non-intersecting lines *within* the circle. Each point is connected to precisely one other point. See an example in the figure 5 .

The problem is now to derive the number of ways these $2n$ points can be pairwise connected. Obviously, for $n = 1$ we have 2 points, which can only be connected to one another, so $c_1 = 1$. For $n = 2$ one has 4 points, and $c_2 = 2$. Namely, point 1 can be connected to point 2 or to point 4. Connection of point 1 to point 3 is not allowed, because it intersects the line between points 2 and 4. The points are numbered clockwise. We use the notation $c_0 = 1$.

[1] *2 points* Let there be a line connecting point 1 to point $2m$. The remaining points are divided into two groups by this line. On this basis, write down a recursion formula for c_n for general n.

[2] *2 points* Use the definition of the so-called generating function

$$P(x) = \sum_{k=0}^{\infty} c_k x^k \tag{9}$$

and the recursion found under part **[1]** to derive an equation that $P(x)$ must satisfy. Solve this equation, which yields $P(x)$ as an explicit function of x.

[3] *2 points* Using this solution, obtain the first few terms in the Taylor expansion of $xP(x) = \sum_k a_k x^k$. Derive the ratio a_k/a_{k-1} for general k. Write the similar ratio c_n/c_{n-1} as a function of n.

[4] *2 points* Give c_n as an explicit function of n.

[5] *2 points* The corresponding contribution ΔS to the entropy of a system is $\Delta S = k_B ln(c_n)$ where k_B is Boltzmann's constant. How does ΔS, in leading order, depend on n in the limit of large n?

8 - Glaciers and climate change

Michiel Helsen, Utrecht University

As a consequence of climate fluctuations, glaciers vary in length. We can study the sensitivity of glaciers to climate change with a simple model.

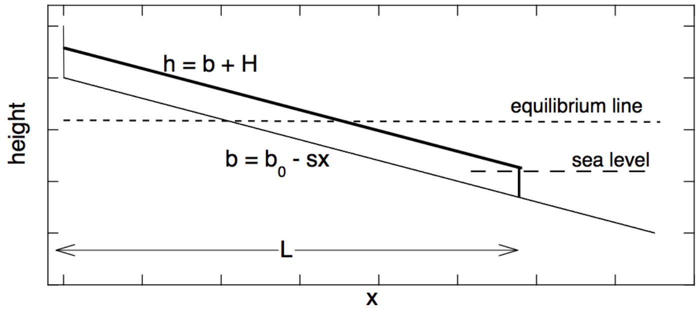

Figure 6: A sketch of a glacier with constant slope

Figure 6 considers a glacier with length L, constant thickness H and flowing on a bed $b(x)$ with a small constant slope s: $b = b_0 - sx$ with $x = 0$ the top of the glacier. The height of the bed h is expressed with respect to sea level.
We assume that the glacier has a constant width. Water and ice density are denoted by ρ_{water} and ρ_{ice}, respectively.
When the glacier reaches the ocean, calving occurs as the ice reaches flotation, i.e. there is no floating ice tongue.

[1] *2 points* Find the length of the part of the glacier extending into the ocean, note that L is the glacier length in x-direction (not along the bed).

Besides floatation the calving rate at the glacier front is proportional to the water depth, which we write as: $c\ b_L$. Apart from ice loss at the calving front, the glacier gains or loses mass at its surface, which is called the Specific Mass Balance *SMB*.
It is defined as the mass of ice that accumulates or is removed per year at a specific point on the glacier surface. As such, it is the resultant of many meteorological processes that determine the exchange of mass between atmosphere and glacier surface (snowfall, rime, melt, sublimation, etc).
We assume that the specific balance varies as: $SMB(h) = \beta(h - E)$, where h is the height above sea level, β is the balance gradient (a constant) and E is the equilibrium line altitude, which separates the accumulation area from the ablation area.

[2] *3 points* Find the solution(s) for the equilibrium length of the glacier, as a function of E.

[3] *1 point* Ice flows under the influence of gravity.
Considering that the glacier geometry is in equilibrium, at which point do we find the highest ice velocity?

Assume for part **[4]** the simple case that the glacier terminus does not reach the ocean.

[4] *3 points* The mean temperature of the atmosphere decreases with height. Assume that E coincides with an isotherm in the atmosphere. Find an expression for the sensitivity of the glacier for a temperature change, i.e. $\frac{dL}{dT}$. The atmospheric temperature gradient is a constant γ.

[5] *1 point* How does this sensitivity changes when temperatures drop and the glacier front reaches the ocean? Show this qualitatively in a sketch.

9 - 24 Hours in a Day – are there?

Gerhard Blab, Utrecht University

We all know that there are 24 hours in a day. If we look more closely, it turns out, this is not quite correct: a true solar day in late December is up to half a minute longer than the expected 24 hours, while in mid-September we are all shortchanged 20 seconds! Only averaged over a year, a "mean" solar day measures the regulation 24 hours.

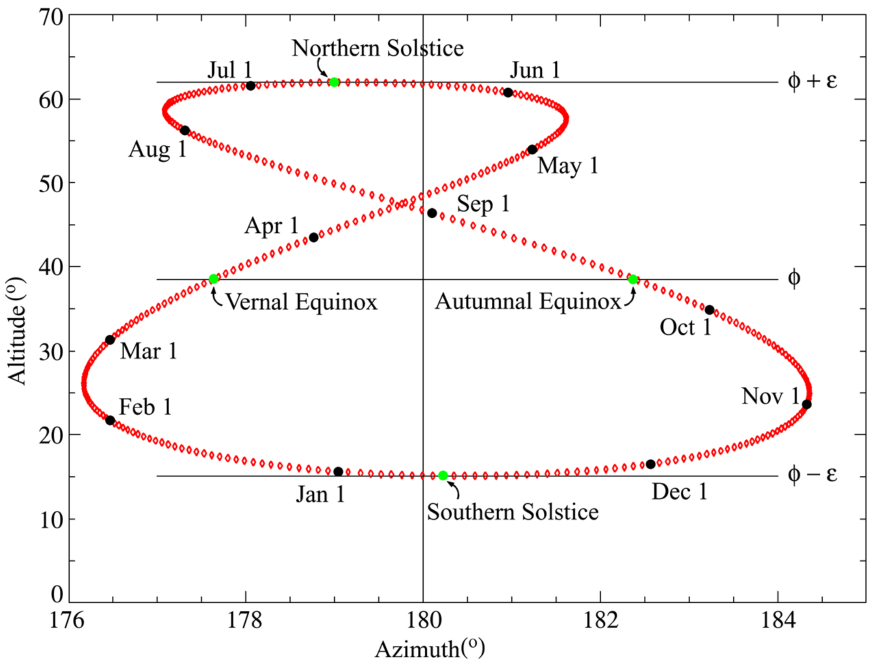

Figure 7: Position of the sun at 12:00 noon GMT as seen at the Royal Observatory, Greenwich, UK (lat $51.5°$ N, long $0°$ W); Earth's last perihelion (147 Gm) occurred on January 4th, 2014, and its next aphelion (152 Gm) will be on July 4th.

To tackle this problem, I need to tell you what I mean by a "true" solar day: it is the time between two local noons, that is times at which your shadow will point exactly north (or south, if you are in Australia).

[1] *3 points* Argue and sketch how the orbital motion of Earth around the sun leads to a mean solar day that is longer than the rotation of the Earth, and show that this difference should indeed be on the order of 4 minutes. Make sure to clearly indicate directions of rotation!

[2] *3 points* Figure 7 shows an "Analemma", a figure obtained by plotting the position of the sun every 24 hours over the course of a year. In it you can find back the seasonal change of altitude, as well as offset between our 24 hour "mean solar day" and the true solar time. Use the figure to estimate the four times during a year at which a day is actually close to

24 hours long, and plot the offset as a function of date (y axis: offset in minutes; x axis: months). This representation is also called the “equation of time”.

[3] *4 points* The equation of time is caused by two different effects, both comparable in magnitude: the eccentricity of Earth’s orbit and the obliquity (“tilt”) of its axis. Explain how those two effects influence the length of a true solar day during a year, and sketch their independent contributions to the equation of time.

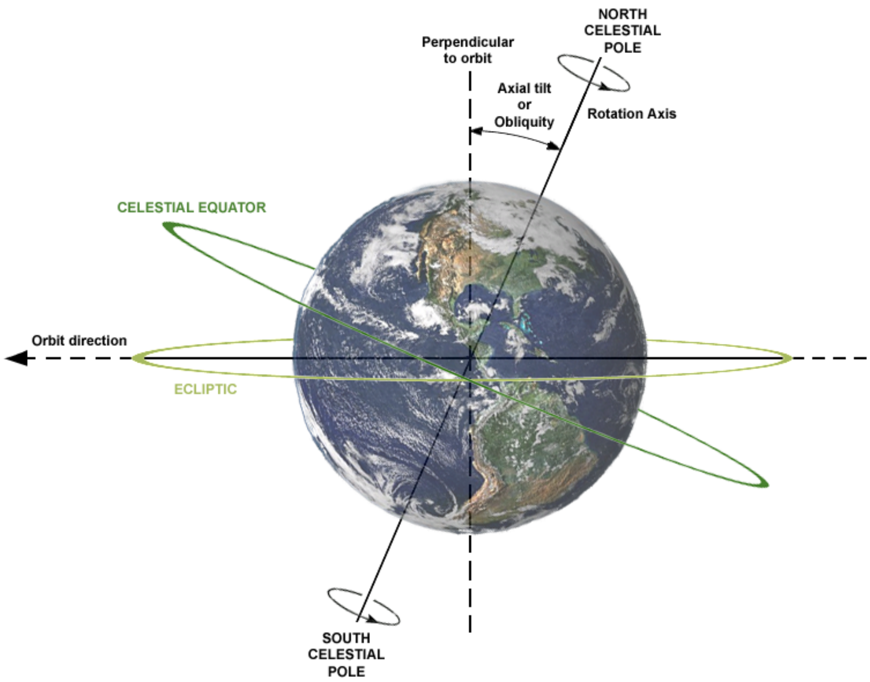

Figure 8: Axis Tilt (Obliquity) of Earth

10 - Dzyaloshinskii-Moriya interactions and skyrmions

Rembert Duine, Utrecht University

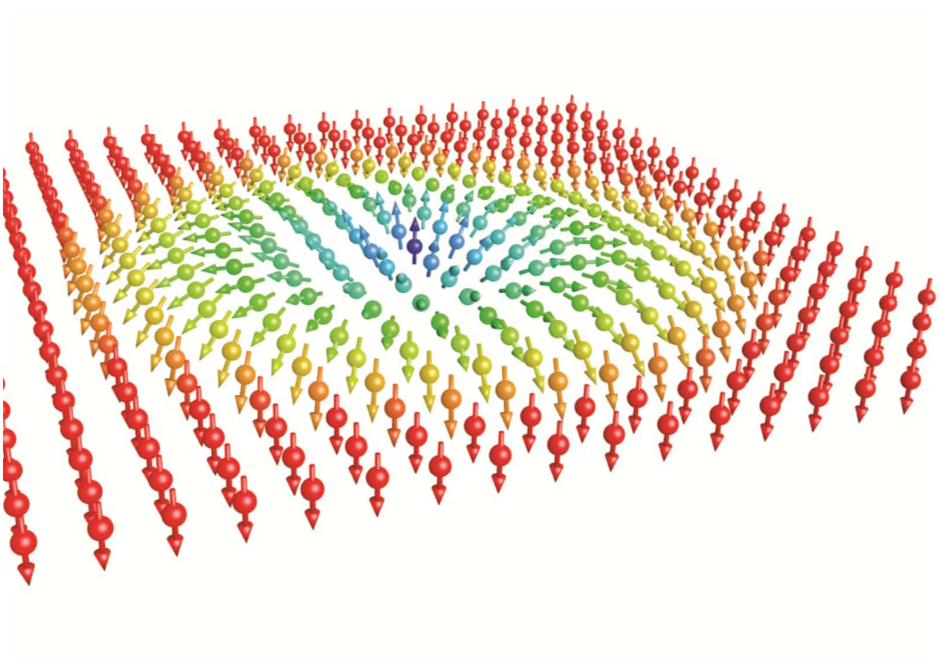

Figure 9: Example of a magnetic skyrmion. The magnetization points up at the skyrmion core and down away from the core.

A ferromagnetic material far below its critical temperature is characterized by a direction of magnetization $\mathbf{m}(\mathbf{x})$ (a unit vector) that can, in general, be a function of the three-dimensional position $\mathbf{x}$. For ordinary ferromagnets, the energy is given by

$$E[\,\mathbf{m}] = \int d\mathbf{x} \left[-\frac{J}{2}\mathbf{m}(\mathbf{x}) \cdot \nabla^2 \mathbf{m}(\mathbf{x}) \right] \qquad (10)$$

where $J > 0$ is the so-called spin stiffness.

[1] *2 points* Give the configuration of $\mathbf{m}(\mathbf{x})$ with the lowest energy.

In certain ferromagnets (to be more precise, in ferromagnets without inversion symmetry) there are additional terms in the energy, called Dzyaloshinskii-Moriya interactions. One possibility is a term of the form $\mathbf{m} \cdot \nabla \times \mathbf{m}$, so that the energy now reads

$$E[\mathbf{m}] = \int d\mathbf{x} \left[-\frac{J}{2}\mathbf{m}(\mathbf{x}) \cdot \nabla^2 \mathbf{m}(\mathbf{x}) + \frac{D}{2}\mathbf{m}(\mathbf{x}) \cdot (\nabla \times \mathbf{m}(\mathbf{x})) \right] \qquad (11)$$

with $D > 0$ a constant.

[2] *2 points* Derive the Euler-Lagrange equations for $\mathbf{m}(\mathbf{x})$ by minimizing this energy functional, and show that you obtain

$$J\nabla^2\mathbf{m}(\mathbf{x}) = D\nabla \times \mathbf{m}(\mathbf{x}) \tag{12}$$

Show that a possible solution of this equation is a so-called spin spiral:

$$\mathbf{m}(\mathbf{x}) = \cos(qx)\hat{y} + \sin(qx)\hat{z} \tag{13}$$

and determine q.

Other examples of low-energy configurations are so-called skyrmions (see figure 9) for which the magnetization depends only on the coordinates in the $x-y-$plane, i.e., $\mathbf{m}$(x) = $\mathbf{m}$(x, y). Skyrmions correspond to excitations where the magnetization is up (or down) at a certain position (the position of the skyrmion), whereas it is down (or up) away from this position. For skyrmions the so-called winding number W is equal to +1 (or -1 for anti-skyrmions). This winding number is defined by

$$W = \int \frac{dxdy}{(4\pi)}\mathbf{m} \cdot \left(\frac{\partial \mathbf{m}}{\partial x}\right) \times \left(\frac{\partial \mathbf{m}}{\partial y}\right) \tag{14}$$

[3] *3 points* Show that this winding number is an integer.
(This means that smooth evolutions of the magnetization cannot lead to changes in the winding number, so that skyrmions are what is called "topologically protected").

Hint: one aproach is to parameterize the unit vector $\mathbf{m}$ in terms of angles $\theta(\rho, \phi)$ and $\varphi(\rho, \phi)$ according to $\mathbf{m} = (sin(\theta)cos(\varphi), sin(\theta)sin(\varphi), cos(\theta))$. These angles depend polar coordinates (ρ, ϕ) in the plane. Rewrite the winding number in terms of ρ and ϕ. Evaluate this for skyrmions for which ρ depends on ϕ only.

A possible ansatz for the description of a skyrmion is to assume

$$\mathbf{m} = \sin\theta(\rho)\hat{\phi} + \cos\theta(\rho)\hat{z} \tag{15}$$

where (ρ, ϕ, z) are cylindrical coordinates.

Continued on next page

[4] *3 points* Derive the equation of motion for $\theta(\rho)$, starting from the energy that includes a field in the z-direction, i.e., starting from

$$E[\mathbf{m}] = \int d\mathbf{x} \left[-\frac{J}{2}\mathbf{m}(\mathbf{x}) \cdot \nabla^2 \mathbf{m}(\mathbf{x}) + \frac{D}{2}\mathbf{m}(\mathbf{x}) \cdot (\nabla \times \mathbf{m}(\mathbf{x})) - B\mathbf{m}(\mathbf{x}) \cdot \hat{z} \right] \tag{16}$$

where B is the magnitude of the field in appropriate units.

You might find the cylindrical Laplacian usefull

$$\nabla^2 = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{17}$$

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