# PLANCKS 2015

## Physics League Across Numerous Countries for Kick-ass Students

# Answers

# Contents

# 1. A Positron in an Electric Field

**(1.1)** [*1 point*] Initial momentum in the $y$-direction is conserved:

$$p_0 = \frac{mu_0}{\sqrt{1-(u_0/c)^2}} = \frac{mu_y}{\sqrt{1-(u_x^2+u_y^2)/c^2}};$$

the momentum in the $x$-direction grows according to

$$p_x = eE_0t = \frac{mu_x}{\sqrt{1-(u_x^2+u_y^2)/c^2}};$$

from these two equations follows straightforwardly that

$$p_0^2 + (eE_0t)^2 = \frac{m^2u^2}{\left(1-(u/c)^2\right)},$$

so that:

$$\gamma(t) = \frac{1}{\sqrt{1-(u/c)^2}} = \sqrt{1+\frac{(eE_0t)^2+p_0^2}{m^2c^2}}.$$

**(1.2)** [*1 point*]

$$u_x(t) = \frac{eE_0t}{\gamma(t)m} = \frac{eE_0t/m}{\sqrt{1+\frac{(eE_0t)^2+p_0^2}{m^2c^2}}}$$

and

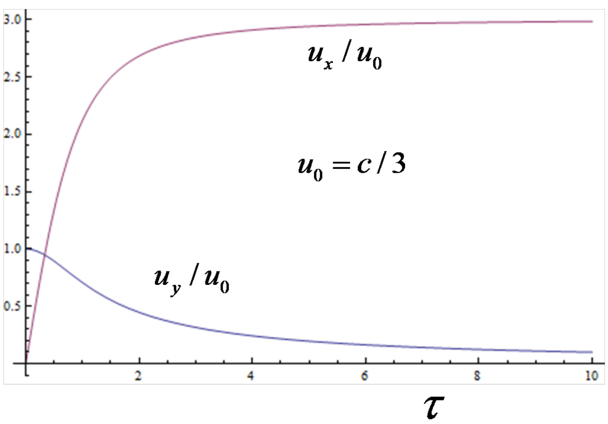
$$u_y(t) = \frac{p_0}{\gamma(t)m} = \frac{p_0/m}{\sqrt{1+\frac{(eE_0t)^2+p_0^2}{m^2c^2}}}$$

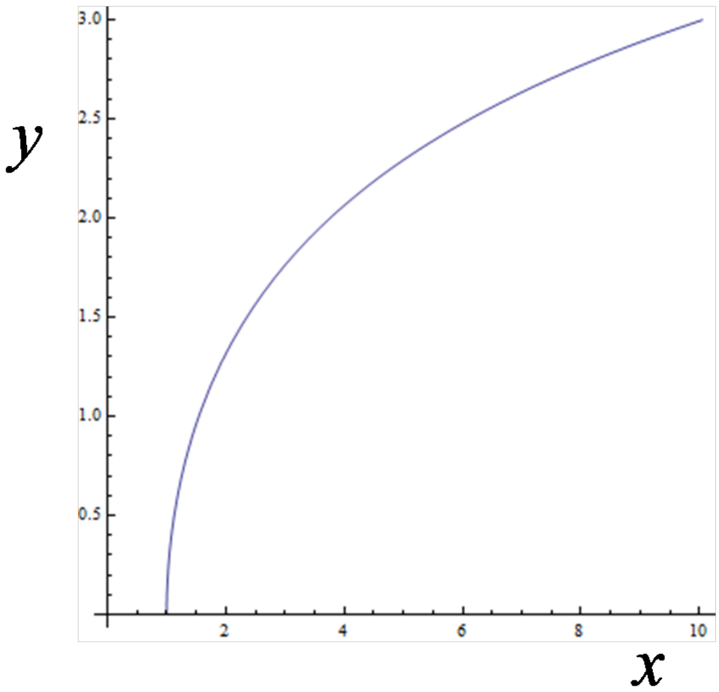
**(1.3)** [*1 point*] using $\tau \equiv \frac{eE_0t}{\sqrt{p_0^2+m^2c^2}}$ and $mu_0 = \frac{p_0}{\sqrt{1+\frac{p_0^2}{m^2c^2}}}$ it follows that

$$\frac{u_x(\tau)}{u_0} = \frac{\tau}{\sqrt{1+\tau^2}}\frac{c}{u_0}$$

and

$$\frac{u_y(\tau)}{u_0} = \frac{1}{\sqrt{1+\tau^2}}\frac{c}{u_0}$$






**(1.4)** [*1 point*] The velocity in the $x$-direction goes asymptotically to the speed of light. Due to conservation of relativistic momentum, the velocity in the $y$-direction goes down, contrary to Newtonian intuition. Above you can see the trajectory of the positron.

**(1.5)** [*2 points*] Use conservation of potential energy into kinetic energy:

$$
\begin{aligned}
&2 \cdot \frac{1}{2} m \left( \dot{x}(t)^2 - \dot{x}(0)^2 \right) = m\dot{x}(t)^2 = \frac{e^2}{4\pi\varepsilon_0 \cdot 2x(0)} - \frac{e^2}{4\pi\varepsilon_0 \cdot 2x(t)} = \frac{e^2}{8\pi\varepsilon_0} \left( \frac{1}{x_0} - \frac{1}{x} \right) \\
&\Rightarrow \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{e^2}{8\pi m\varepsilon_0} \left( \frac{1}{x_0} - \frac{1}{x} \right)} \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \sqrt{\frac{8\pi m\varepsilon_0 x_0}{e^2}} \left( 1 - \frac{x_0}{x} \right)^{-\frac{1}{2}} \\
&\Rightarrow \frac{\mathrm{d}(t/t_0)}{\mathrm{d}(x/x_0)} = \left( 1 - \frac{x_0}{x} \right)^{-\frac{1}{2}},
\end{aligned}
$$

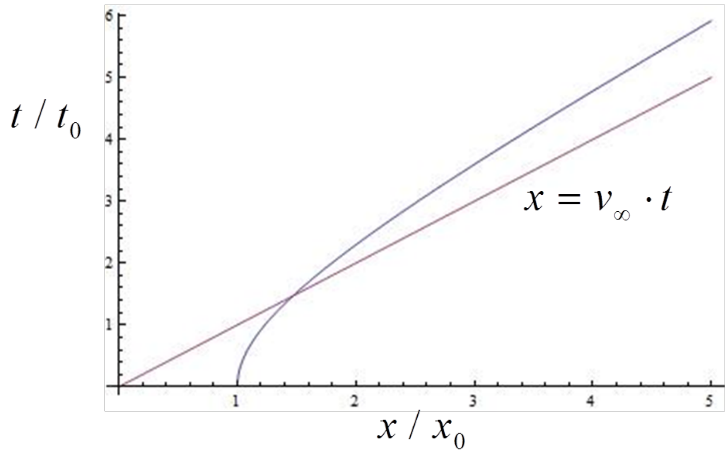
with $t_0 = \sqrt{\dfrac{8\pi m\varepsilon_0 x_0^3}{e^2}}$.

Using as boundary condition $x(t=0) = x_0 \Rightarrow t(x_0) = 0$, it can be checked straightforwardly that

$$
\frac{t}{t_0} = \sqrt{\left( \frac{x}{x_0} - 1 \right) \frac{x}{x_0}} + \ln \left( \frac{x}{x_0} + \sqrt{\frac{x}{x_0} - 1} \right)
$$

**(1.6)** [*2 points*] The asymptotic speed follows from

$$
\begin{aligned}
\lim_{x\to\infty} \frac{t}{x} &= t_0 \lim_{x\to\infty} \frac{1}{x} \left[ \sqrt{\left( \frac{x}{x_0} - 1 \right) \frac{x}{x_0}} + \ln \left( \frac{x}{x_0} + \sqrt{\frac{x}{x_0} - 1} \right) \right] \\
&= \lim_{x\to\infty} \frac{t_0}{x} \left( \frac{x}{x_0} + \ln \left( 2\frac{x}{x_0} \right) \right) = \frac{t_0}{x_0} \\
&\Rightarrow v_\infty = \lim_{x\to\infty} \frac{x}{t} = \frac{x_0}{t_0}
\end{aligned}
$$




Non-relativistic is justified if

$$v_\infty = \frac{x_0}{t_0} = \frac{e}{\sqrt{8\pi\varepsilon_0 m x_0}} \ll c \Rightarrow 2x_0 \gg \frac{e^2}{4\pi\varepsilon_0 mc^2} = 2.8 \times 10^{-15} \text{ m},$$

i.e. if the initial separation is much larger than the classical electron radius. In practice this is always the case.

**(1.7)** [*2 points*] The electric field vector $\vec{E}^{\text{com}}_{2\to1}$ in the center-of-mass frame is perpendicular to the direction of acceleration, so in the lab frame the electric field strength $\vec{E}^{\text{lab}}_{2\to1}$ is magnified by the Lorentz factor:

$$\vec{E}^{\text{lab}}_{2\to1} = \gamma \vec{E}^{\text{com}}_{2\to1} = -\gamma \frac{e}{4\pi\varepsilon_0 (2x_1)^2}\hat{x}.$$

The magnetic field vector in the lab frame follows from:

$$\vec{B}^{\text{lab}}_{2\to1} = \gamma\left(\vec{v} \times \frac{\vec{E}^{\text{com}}_{2\to1}}{c^2}\right) = \gamma\left(v_z \frac{-e}{4\pi\varepsilon_0(2x_1)^2}\frac{1}{c^2}\hat{z}\times\hat{x}\right) = \gamma\frac{-v_z e}{4\pi\varepsilon_0 c^2 (2x_1)^2}\hat{y}.$$

Note the minus sign in front of $\vec{v}$, which is due to proper application of the Lorentz transformation of the fields.

**(1.8)** [*2 points*] The Lorentz force in the lab frame is given by

$$\begin{aligned}
\vec{F}^{\text{lab}}_{2\to1} &= -e\left(\vec{E}^{\text{lab}}_{2\to1} + \vec{v}\times\vec{B}^{\text{lab}}_{2\to1}\right)\\
&= \gamma\frac{e^2}{4\pi\varepsilon_0(2x_1)^2}\hat{x} + \gamma\frac{v_z^2 e^2}{4\pi\varepsilon_0 c^2(2x_1)^2}\hat{z}\times\hat{y} + \gamma\frac{v_x v_z e^2}{4\pi\varepsilon_0 c^2 (2x_1)^2}\hat{x}\times\hat{y}\\
&= \gamma\frac{(1-v_z^2)e^2}{4\pi\varepsilon_0(2x_1)^2}\hat{x} + \gamma\frac{v_x v_z e^2}{4\pi\varepsilon_0 c^2(2x_1)^2}\hat{z}.
\end{aligned}$$

Note that the transverse motion gives rise to acceleration of the beam as well. The force in the transverse direction may be approximated by

$$F^{\text{lab}}_{2\to1} = \gamma\frac{(1-v_z^2)e^2}{4\pi\varepsilon_0(2x_1)^2} \approx \frac{(1-v^2)e^2}{4\pi\varepsilon_0(2x_1)^2} = \frac{1}{\gamma}\frac{e^2}{4\pi\varepsilon_0(2x_1)^2}.$$

By assumption, the motion in the $x$-direction is non-relativistic, i.e. $\dot{p}_x \approx \gamma m\ddot{x}$, so the equation of motion in the transverse direction can be approximated by $\gamma m\ddot{x} \approx F^{\text{lab}} \approx \frac{1}{\gamma}F^{\text{com}}_x$, and therefore $m\ddot{x} \approx \gamma^{-2}F^{\text{com}}_x$. The suppression with a factor $1/\gamma^2$ can be explained as follows: one factor $\gamma$ can be attributed to slowing down of the repulsion due to time dilation, and the other factor $\gamma$ to the relativistic increase in mass.

# 2. Configurations of DNA Molecule

**(2.1)** [*3 points*] The two straight lines have length $l$ each, the curved piece arclength $s$ and curvature $\frac{1}{r}$. $l$ and $r$ are related to the apex angel$\alpha$ via $\frac{l}{r} = cot(\frac{\alpha}{2})$. The total lenght $L$ is given by

$$L = 2l + s = 2l + (\pi + \alpha)r = 2rcot(\frac{\alpha}{2}) + (\pi + \alpha)r.$$

The bending energy is given by

$$E = \frac{A}{2}\frac{s}{r^2} = \frac{A}{2}\frac{\pi + \alpha}{r} = \frac{A}{2L}(\pi + \alpha)\left[2cot(\frac{\alpha}{2}) + \pi + \alpha\right]$$

where $A$ denotes the bending modulus. Minimization with respect to $\alpha$ leads to

$$\pi + \alpha + 2cot(\frac{\alpha}{2}) + (\pi + \alpha)\left[1 - \frac{1}{sin^2(\frac{\alpha}{2})}\right] = 0$$

This can be simplified to

$$-\frac{1}{sin^2(\frac{\alpha}{2})}((\pi + \alpha)cos\alpha - sin\alpha) = 0$$

This leaves us with solving the transcendental equation $\pi + \alpha = tan\alpha$. This is solved for $\alpha = 77.5°$

**(2.2)** [*3 points*]

$$\left\langle R^2\right\rangle = \left\langle \mathbf{R}^2\right\rangle = \left\langle (\int_0^L \mathbf{t}(s)ds)^2\right\rangle = \int_0^L ds\int_0^L ds'\left\langle \mathbf{t}(s)\cdot\mathbf{t}(s')\right\rangle = \int_0^L ds\int_0^L ds' e^{-\frac{|s-s'|}{l_p}} = 2l_p^2(\frac{L}{l_p} + e^{-\frac{L}{l_p}} - 1)$$

**(2.3)** [*3 points*]

$$\left\langle R^2\right\rangle = \left\langle \mathbf{R}^2\right\rangle = \left\langle (\sum_{i=1}^N \mathbf{r}_i)^2\right\rangle = \sum_{i,j=1}^N \langle \mathbf{r}_i\cdot\mathbf{r}_j\rangle = \sum_{i=1}^N \langle \mathbf{r}_i\cdot\mathbf{r}_i\rangle + \sum_{i\neq j}\langle \mathbf{r}_i\cdot\mathbf{r}_j\rangle = \sum_{i=1}^N \langle \mathbf{r}_i\cdot\mathbf{r}_i\rangle = b^2N$$

**(2.4)** [*3 points*] The expression for $\left\langle R^2\right\rangle$ of long wormlike chains with $L \gg l_p$ can be approximated by

$$\left\langle R^2\right\rangle = 2l_pL.$$

This can be interpreted (up to a numerical factor) as a flexible chain of segment lenght $l_p$ with a total number of segments $N = \frac{L}{l_p}$ which according to 2.3 has a mean-squared radius

$$\left\langle R^2\right\rangle = b^2N = l_p^2\frac{L}{l_p} = l_pL$$

(if you care to have the prefactor right: choose $b = 2l_p$).

# 3. Falling Slinky

**(3.1)** [*3 points*] The pulling force in each part of the slinky is determined by its **relative local extension**, such that Hooke's law can be written as $F(x) = k\frac{dL}{dx}$. As this force has to lift the lower part of the slinky $F(x) = mgx = k\frac{dl}{dx}$. Integration of this equation yields the shape of the slinky $L(x) = \left[\frac{mg}{2k}\right] x^2$, which corresponds to a quadratic 'density profile' of its winding. The total length of the slinky is $L_0 = \frac{mg}{2k}$ is needed for later reference.

**(3.2)** [*1 point*] When the top part of the slinky is falling, **the bottom part doesn't notice yet that its local shape hasn't changed yet**. The acceleration of the top part is faster than that of a free-falling object. as this top experiences the pulling force of the lower part.

**(3.3)** [*5 points*] In order to calculate how long it takes for the top of the slinky to reach the bottom of the slinky, you don't need to solve its full equation of motion. It is enough to **consider the motion of the center of mass of the slinky**, which is originally positioned at $\frac{L_0}{3}$ above the bottom of the slinky. This center of mass moves as any free falling object does and reaches the bottom at a time $t_{\text{fall}}$ that obeys the equation $\frac{L_0}{3} = \frac{g}{2}t_{\text{fall}}^2$. This yields $t_{\text{fall}} = \sqrt{\frac{2L_0}{3g}}$, which is a factor $\sqrt{3}$ shorter than the fall time $\sqrt{\frac{2L_0}{g}}$ of a point-like object falling over a distance $L_0$.

**(3.4)** [*3 points*] We can derive an equation for the distance $\Delta L(t)$ travelled by the top of the slinky at a time $t$ after 'launch', up to the moment when it reaches the bottom of the slinky, by combining the equation for the motion of the center of mass with the observation that a compression wave travels neatly from top to bottom through the slinky. When at a time $t$ a fraction $y = l - x$ of the top of the slinky has collapsed, the position of the center of mass with respect to the bottom of the slinky can be written as $\frac{L_0}{3} - \frac{1}{2}gt^2 = L_0(x^2 - \frac{2}{3}x^3)$. This equation describes how the slinky contracts in time, but the solution $x(t)$ is far from trivial. Its time derivative yields the expression $\frac{dx}{dt} = \frac{-gt}{2L_0x(1-x)}$ and the real speed $v = -\frac{dL}{dt} = -2L_0x\frac{dx}{dt} = \frac{gt}{1-x}$. As an alternative approach towards the solution, we can consider the acceleration of the contracted top section of the slinky, which goes as $\frac{dv}{dt} = \frac{g}{1-x}$.

# 4. Measuring Interlayer States in Graphene and Graphite

**(4.1)** [*1 point*]

$$k = \frac{2\pi}{h}\sqrt{2mE}$$

$$\Rightarrow \lambda = \frac{2\pi}{k} = \frac{h}{\sqrt{2mE}}$$

For $E = 5$ eV, we find $\lambda = 5.5 \times 10^{-10}$ m$= 0.55$ nm $\Rightarrow$ lateral resolution $r = 2.6\lambda$.

This is good but there is still room for improvement, at least up to $r = \lambda = 0.55$ nm.

**(4.2)** [*1 point*] At $E \to 0$ eV, we have $\lambda \to \infty$. Hence the resolution will get extremely bad. Physically, this situation means that the electrons just do not reach the sample.

**(4.3)** [*2 points*] $\Psi_1(z)$ and $\Psi_2(z)$ are degenerated, both with eigenenergy $\varepsilon$ i.e.

$$\left\langle \Psi_1 | H | \Psi_1 \right\rangle = \varepsilon = \left\langle \Psi_2 | H | \Psi_2 \right\rangle$$

Furthermore

$$\left\langle \Psi_1 | H | \Psi_2 \right\rangle = \left\langle \Psi_2 | H | \Psi_1 \right\rangle = -t$$

The latter leads to off-diagonal terms: if we write $H$ in matrix form in basis $\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ we have

$$H = \begin{pmatrix} \varepsilon & -t \\ -t & \varepsilon \end{pmatrix}$$

**(4.4)** [*2 points*] To find the eigenvalues we require that

$$det\,(H - E \cdot I) = 0 \Leftrightarrow det \begin{pmatrix} \varepsilon - E & -t \\ -t & \varepsilon - E \end{pmatrix} = 0$$

$$\to (\varepsilon - E)^2 - t^2 = 0$$

$$E_{\pm} = \varepsilon \pm t$$

For $E_+ = \varepsilon + t$, we have eigenfunction (the odd one)

$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}\left(|Psi_1\rangle - |Psi_2\rangle\right)$$

For $E_- = \varepsilon - t$, we have the even eigenfunction:

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left(|Psi_1\rangle + |Psi_2\rangle\right)$$

**(4.5)** [*1 point*] See figure 1.

**(4.6)** [*2 points*] Graphite: $N$ interlayer states with $N >> 1$. We use that also:

$$\langle k_z| = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} e^{-imk_zc} \langle \Psi_m|$$

We want to know

$$\left\langle k_z | H | k_z \right\rangle = \frac{1}{N} \sum_{m=1}^{N} \sum_{n=1}^{N} e^{ik_zc(n-m)} \left\langle \Psi_m | H | \Psi_n \right\rangle$$

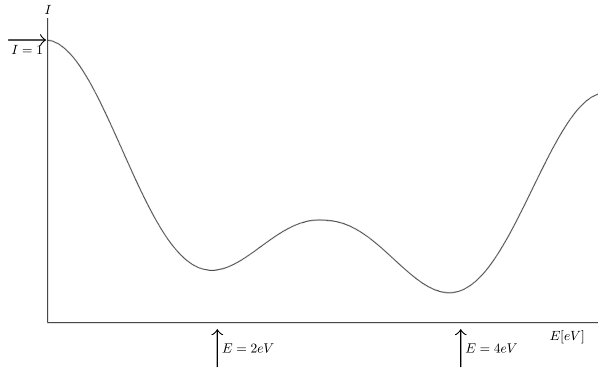



Figure 1: Plot of Intensity vs. Energy. Notice the maximum at E=0 and the two minima at E=2eV and E=4eV.

Now all hopping terms vanish except if $m = n - 1$ or $m = n + 1$ (these give $\langle \Psi_m | H | \Psi_n \rangle = -t$) or if $m = n$ (this gives $\varepsilon$). Thus

$$\langle k_z | H | k_z \rangle = \frac{1}{N} \sum_{n=1}^{N} (e^{-ik_z c} + e^{ik_z c})(-t) + \frac{1}{N} \sum_{n=1}^{N} e^0 \varepsilon$$

$$\Rightarrow \langle k_z | H | k_z \rangle = \varepsilon - 2t \cos(k_z c)$$

This is (to good approximation) the dispersion relation $\varepsilon(k_z)$

**(4.7)** [*1 point*] See Figure 2.

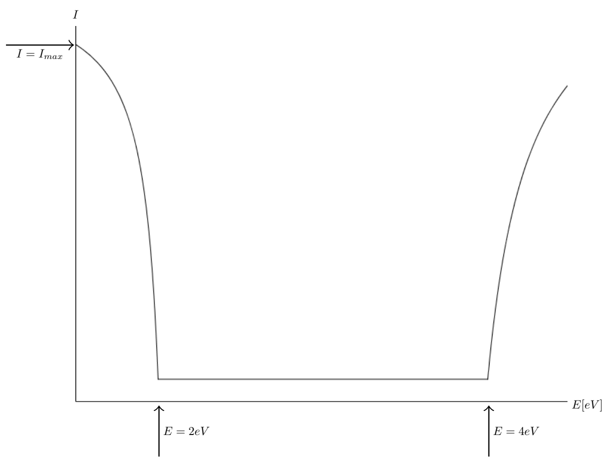


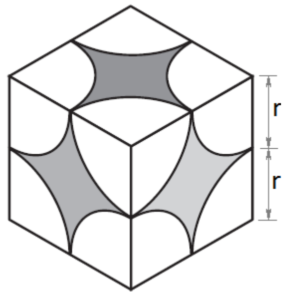

Figure 2: Plot of Intensity vs. Energy. Notice the maximum at E=0 and the two minima at E=2eV and E=4eV.

Because there is a band now, with width 4eV$=4t$ ($2t\cos(k_z c)$ goes from $-2t$ to $2t$) There is an "infinite number" of minima between 1eV and 5eV.

**(4.8)** [*2 points*] If the angle is not normal, only the normal component of the kinetic energy will determine the position of the minimum. Hence more kinetic energy is needed to reach the minimum. In other words, the minimum shifts up in energy.

It turns out that this method (i.e. changing the incident angle) is a way to directly measure also the lateral dispersion relations, i.e. in the $(x, y)$ direction.

# 5. Physics of the Oil and Gas Production




**(5.1)** [*1 point*] Let's cut a cube from the pile of balls, as shown in the figure.

Using the definition of porosity:

$$\phi = \frac{(2r)^3 - 8 \cdot \frac{1}{8}(\frac{4}{3}\pi r^3)}{(2r)^3} = 1 - \frac{\pi}{6} \tag{1}$$

$\phi = 0.476$ or 47.6%

**(5.2)** [*1 point*] If the fluid is incompressible, the fluid velocity along the tube is constant. It varies in the radial direction only, which is due to viscous forces. Let's consider a part of the fluid with a cylindrical shape, with radius $y$, which is coaxial to the cylinder with radius $r_0$. Using the definition of the internal friction, the force balance could be written in the following way:

$$(p_1 - p_2) \cdot \pi y^2 = -2\pi y L_0 \mu \frac{dv}{dy} \tag{2}$$

After rearranging variables

$$\frac{(p_1 - p_2)}{2\mu L_0} \int_{r_0}^{y} y dy = -\int_0^v dv \tag{3}$$

Which results in the velocity distribution:

$$v(y) = \frac{(p_1 - p_2)}{4\mu L_0}({r_0}^2 - y^2) \tag{4}$$

**(5.3)** [*1 point*] The total fluid volume flowing through the tube in a unit of time:

$$q = \int_0^{r_0} v \cdot 2\pi y dy = \frac{(p_1 - p_2)\pi}{2\mu L_0} \int_0^{r_0} ({r_0}^2 y - y^3) dy = \frac{(p_1 - p_2)\pi {r_0}^4}{8\mu L_0} \tag{5}$$

Comparing with the Poiseuille equation gives $k_0 = \frac{{r_0}^2}{8}$

**(5.4)** [*1 point*] For estimations one can assume that a porous medium could be modeled as tubes with the radii equal to the size of the balls:

$$k \approx \frac{{r_0}^2}{8}(1 - \frac{\pi}{6})^2 \approx 3 \cdot 10^{-14} m^2 \tag{6}$$

**(5.5)** [*1 point*] From the law of conservation of mass and the condition that the fluid is incompressible can be concluded that the flow rate is constant everywhere:

$$q = \frac{k_1}{\mu} A \frac{P_{in} - P_b}{L_1} = \frac{k_2}{\mu} A \frac{P_b - P_{out}}{L_2} \tag{7}$$

$$P_b = \frac{\frac{k_1}{L_1} P_{in} + \frac{k_2}{L_2} P_{out}}{\frac{k_1}{L_1} + \frac{k_2}{L_2}} \tag{8}$$

**(5.6)** [*1 point*] Using eq.(8)

$$q = \frac{k_{eff}}{\mu} A \frac{P_{in} - P_{out}}{L_1 + L_2} = \frac{A}{\mu} \frac{k_1}{L_1} \left[ P_{in} - \frac{\frac{k_1}{L_1} P_{in} + \frac{k_2}{L_2} P_{out}}{\frac{k_1}{L_1} + \frac{k_2}{L_2}} \right] \tag{9}$$

$$k_{eff} = \frac{k_1 k_2 (L_1 + L_2)}{k_1 L_2 + k_2 L_1} \tag{10}$$

**(5.7)** [*1 point*]

$$v_w = \frac{q}{\pi {r_w}^2} = \frac{30/86400}{3.14 \cdot 0.1^2} = 1.1 \cdot 10^{-2} m/s \tag{11}$$

$$v_{res} = \frac{q}{2\pi r_w h} = \frac{30/86400}{2 \cdot 3.14 \cdot 0.1 \cdot 10} = 5.5 \cdot 10^{-5} m/s \tag{12}$$

The velocity at the reservoir is very small and it depends on the definition of the fluid velocity in the reservoir. The fluid particles actually move a few orders of magnitude faster than Darcy's velocity $\approx \frac{v_{res}}{\phi}$.

**(5.8)** [*1 point*] For the incompressible fluid:

$$q = \frac{k}{\mu} 2\pi r h \frac{dP}{dr} = const \tag{13}$$

Changing variables:

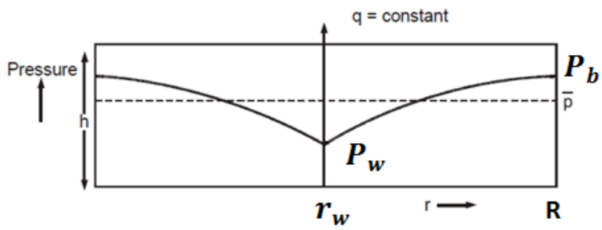
$$\int_{P_w}^{P} dP = \frac{q\mu}{2\pi k h} \int_{r_w}^{r} \frac{dr}{r} \tag{14}$$

$$P - P_w = \frac{q\mu}{2\pi k h} ln\left(\frac{r}{r_w}\right) \tag{15}$$

Finally:

$$P_b - P_w = \frac{q\mu}{2\pi k h} ln\left(\frac{R}{r_w}\right) \tag{16}$$

**(5.9)** [*1 point*]




**(5.10)** [*1 point*]

$$q = \frac{dV_{fluid}}{dt} = \frac{d(\phi V_{res})}{dt} = \phi V_{res} \left(\frac{1}{V_{res}} \frac{dV_{res}}{d\bar{p}}\right) \frac{d\bar{p}}{dt} = -2\phi L^2 h c_r \frac{d\bar{p}}{dt} \tag{17}$$

$$\alpha = -2\phi L^2 h c_r \tag{18}$$

**(5.11)** [*2 points*] Applying Darcy's law:

$$q = 2\frac{k}{\mu} L h \frac{P_b - P_w}{L} = -2\phi L^2 h c_r \frac{d\bar{p}}{dt} \tag{19}$$

The average pressure can be estimated as:

$$\bar{p} \approx \frac{p_b + p_w}{2} \tag{20}$$

considering that $P_w$ is constant:

$$d\bar{p} \approx \frac{1}{2} dP_b \tag{21}$$

Using Eq.(20) and (21):

$$\int_{P_b(0)}^{P_b(t)} \frac{dP_b}{P_b - P_w} = -\frac{2k}{\mu L^2 c_r \phi} \int_0^t dt \tag{22}$$

$$ln\left(\frac{P_b(t) - P_w}{P_b(0) - P_w}\right) = -\frac{2k}{\mu L^2 c_r \phi} t \tag{23}$$

Using the definition of the flow rate from (21):

$$ln\left(\frac{q(t)}{q_0}\right) = -\frac{2k}{\mu L^2 c_r \phi} t \tag{24}$$

$$q(t) = q_0 exp\left(-\frac{2k}{\mu L^2 c_r \phi} t\right) \tag{25}$$

# 6. Scattering

**(6.1)** [*2 points*] The Yukawa potential is spherically symmetric, so the amplitude is a function of $|\mathbf{k}-\mathbf{k}'| \equiv q = 2k\sin\frac{\theta}{2}$. Performing the angular integrals in

$$f^{(1)}(\mathbf{k},\mathbf{k}') = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int \mathrm{d}^3x'\; e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}\, V(\mathbf{x}')\;, \tag{1}$$

yields

$$f^{(1)}(\mathbf{k},\mathbf{k}') = f^{(1)}(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty \mathrm{d}r\; r\sin(qr)V(r)\;. \tag{2}$$

Now insert the Yukawa potential and integrate over $r$, using (the imaginary part of)

$$\int_0^\infty \mathrm{d}r\; e^{-(\mu-iq)r} = \frac{1}{\mu-iq}\;, \tag{3}$$

which holds for real $\mu, q > 0$. Plugging $f(\mathbf{k},\mathbf{k}') \simeq f^{(1)}(\mathbf{k},\mathbf{k}')$ into

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\mathbf{k},\mathbf{k}')|^2\;. \tag{4}$$

produces the desired result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \simeq \left(\frac{2mV_0}{\mu\hbar^2}\right)^2 \frac{1}{\big(2k^2(1-\cos\theta)+\mu^2\big)^2}\;. \tag{5}$$

.

**(6.2)** [*2 points*] Coulomb potential. The differential cross section in

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{mZZ'e^2}{|\mathbf{p}|^2}\right)^2 \frac{1}{(1-\cos\theta)^2}\;. \tag{6}$$

is obtained from

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \simeq \left(\frac{2mV_0}{\mu\hbar^2}\right)^2 \frac{1}{\big(2k^2(1-\cos\theta)+\mu^2\big)^2}\;. \tag{7}$$

by letting $\mu \to 0$ while keeping $V_0/\mu = ZZ'e^2$ fixed and subsequently taking the classical limit $\hbar k = |\mathbf{p}|$.

$$\begin{aligned}\frac{\mathrm{d}\sigma_{\mathrm{qm}}}{\mathrm{d}\Omega} &\simeq \left(\frac{2mV_0}{\mu\hbar^2}\right)^2 \frac{1}{\big(2k^2(1-\cos\theta)+\mu^2\big)^2} \\ &\xrightarrow{\mu\to 0} \left(\frac{2mZZ'e^2}{\hbar^2}\right)^2 \frac{1}{\big(2k^2(1-\cos\theta)\big)^2} \\ &\xrightarrow{\hbar k\to|\mathbf{p}|} \left(\frac{mZZ'e^2}{|\mathbf{p}|^2}\right)^2 \frac{1}{(1-\cos\theta)^2}\end{aligned} \tag{8}$$

Applying the same replacements to the Yukawa potential in

$$V(r) = V_0\frac{e^{-\mu r}}{\mu r}\;, \tag{9}$$

yields the classical Coulomb potential,

$$V_0\frac{e^{-\mu r}}{\mu r} \;\longrightarrow\; \frac{ZZ'e^2}{r} = V_{\mathrm{Coulomb}}(r)\;. \tag{10}$$

**(6.3)** [*3 points*] Multiply the Legendre equation by $P_{\ell'}(x)$ and integrate over $x \in [-1, 1]$. After performing integration by parts on the first term, the equation becomes

$$\int_{-1}^{1} \mathrm{d}x \left( (x^2-1)\frac{\mathrm{d}P_\ell(x)}{\mathrm{d}x}\frac{\mathrm{d}P_{\ell'}(x)}{\mathrm{d}x} + \ell(\ell+1)P_\ell(x)P_{\ell'}(x) \right) = 0 \ . \tag{11}$$

The first term is symmetric under interchange of $\ell$ and $\ell'$, but the second term is not. Subtracting the same equation with $\ell$ and $\ell'$ interchanged thus gives the identity

$$\Big(\ell(\ell+1) - \ell'(\ell'+1)\Big) \int_{-1}^{1} \mathrm{d}x P_\ell(x)P_{\ell'}(x) = 0 \ . \tag{12}$$

It follows immediately that $\int_{-1}^{1} \mathrm{d}x P_\ell(x)P_{\ell'}(x) = 0$ for $\ell \neq \ell'$. So we can write

$$\int_{-1}^{1} \mathrm{d}x P_\ell(x)P_{\ell'}(x) = A_\ell \ \delta_{\ell,\ell'} \ . \tag{13}$$

The normalisation factor $A_\ell$ may be determined by direct evaluation.

$$A_\ell = \int_{-1}^{1} \mathrm{d}x P_\ell^2(x) = \left(\frac{1}{2^\ell \, \ell!}\right)^2 \int_{-1}^{1} \mathrm{d}x \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\Big((x^2-1)^\ell\Big) \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\Big((x^2-1)^\ell\Big) \ . \tag{14}$$

Apply integration by parts $\ell$ times. Each time the boundary term vanishes, leaving

$$A_\ell = \frac{(-1)^\ell}{\big(2^\ell \, \ell!\big)^2} \int_{-1}^{1} \mathrm{d}x (x^2-1)^\ell \frac{\mathrm{d}^{2\ell}}{\mathrm{d}x^{2\ell}}\Big((x^2-1)^\ell\Big) = \frac{(2\ell)!}{\big(2^\ell \, \ell!\big)^2} \int_{-1}^{1} \mathrm{d}x (1-x^2)^\ell \ . \tag{15}$$

It is easy to check that $A_0 = 2$. Assume now that $\ell > 0$ and write $(1-x^2)^\ell = (1-x^2)^{\ell-1} + \frac{x}{2\ell}\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)^\ell$. Integration by parts on the second term then leads to the recurrence relation

$$A_\ell = \frac{2\ell-1}{2\ell}A_{\ell-1} - \frac{1}{2\ell}A_\ell \qquad \Longrightarrow \qquad A_\ell = \frac{2\ell-1}{2\ell+1}A_{\ell-1} \ , \tag{16}$$

whose solution is found to be

$$A_\ell = \frac{2\ell-1}{2\ell+1}\,\frac{2\ell-3}{2\ell-1}\,\frac{2\ell-5}{2\ell-3} \cdots \frac{5}{7}\,\frac{3}{5}\,\frac{1}{3}\,2 = \frac{2}{2\ell+1} \ . \tag{17}$$

Together with eq. (13) this result proves the orthogonality property.

**(6.4)** [*3 points*] Let's start with the left hand side of the optical theorem and insert

$$e^{ikz} \simeq \sum_{\ell=0}^{\infty}(2\ell+1)\frac{1}{2ikr}\Big(e^{ikr} - (-1)^\ell e^{-ikr}\Big)P_\ell(\cos\theta) \quad \text{for large } r \ , \tag{18}$$

$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\theta) \ .$$

Using $P_\ell(1) = 1$, as can be obtained from the Rodrigues formula,

$$\mathrm{Im} f(\theta=0) = \sum_{\ell=0}^{\infty}(2\ell+1)\mathrm{Im}\big[f_\ell(k)\big] \ . \tag{19}$$

On the right hand side of the optical theorem we write

$$\sigma_{\text{tot}} = \int \mathrm{d}\Omega \frac{\mathrm{d}\sigma_{\text{tot}}}{\mathrm{d}\Omega} = \int \mathrm{d}\Omega |f(\theta)|^2 \tag{20}$$

Inserting $f(\theta)$ from eq. (19) and using the orthogonality of Legendre polynomials

$$\int_{-1}^{1} \mathrm{d}x\; P_\ell(x)\, P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell,\ell'} \; . \tag{21}$$

we get,

$$\begin{aligned}\sigma_{\rm tot} &= 2\pi \int_{-1}^{1} \mathrm{d}\cos\theta \sum_{\ell,\ell'=0}^{\infty} (2\ell+1)(2\ell'+1) f_\ell(k) f_{\ell'}(k) P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \\ &= 4\pi \sum_\ell (2\ell+1)|f_\ell(k)|^2 .\end{aligned} \tag{22}$$

In terms of the partial amplitudes the optical theorem thus reads

$$\sum_{\ell=0}^{\infty}(2\ell+1)\mathrm{Im}\big[f_\ell(k)\big] = k\sum_{\ell=0}^{\infty}(2\ell+1)|f_\ell(k)|^2 \; . \tag{23}$$

Comparing coefficients we find $\mathrm{Im}\big[f_\ell(k)\big] = k|f_\ell(k)|^2$. Rewrite this identity as

$$\begin{aligned}\frac{f_\ell(k) - f_\ell^*(k)}{2i} &= k f_\ell(k) f_\ell^*(k) \\ 2ik\big(f_\ell(k) - f_\ell^*(k)\big) &= -4k^2 f_\ell(k) f_\ell^*(k) \\ 1 + 2ik\big(f_\ell(k) - f_\ell^*(k)\big) + 4k^2 f_\ell(k) f_\ell^*(k) &= 1 \\ \big(1 + 2ik f_\ell(k)\big)\big(1 - 2ik f_\ell^*(k)\big) &= 1 \\ |1 + 2ik f_\ell(k)|^2 &= 1 \\ |S_\ell(k)|^2 &= 1\end{aligned} \tag{24}$$

The physical interpretation of the condition $|S_\ell(k)|^2 = 1$ is conservation of probability / flux. To see this clearly we may write out the formula for the wave function, clearly separating the incoming and outgoing spherical waves.

$$\begin{aligned}\psi(\mathbf{x}) &\sim e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \\ &\sim \frac{1}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\Big(e^{ikr} - (-1)^\ell e^{-ikr}\Big)P_\ell(\cos\theta) + \frac{e^{ikr}}{r}\sum_{\ell=0}^{\infty}(2\ell+1) f_\ell(k) P_\ell(\cos\theta) \\ &= \frac{1}{2ikr}\sum_{\ell=0}^{\infty}(2\ell+1)\Big(\big[1 + 2ikr f_\ell(k)\big]e^{ikr} - (-1)^\ell e^{-ikr}\Big)P_\ell(\cos\theta) \; .\end{aligned} \tag{25}$$

Conservation of flux dictates that the phases of the incoming spherical wave $e^{-ikr}$ and the outgoing spherical wave $e^{ikr}$ must have the same modulus: $|1 + 2ik f_\ell(k)|^2 = |-(-1)^\ell|$, or $|S_\ell(k)|^2 = 1$.

**(6.5)** [*2 points*] If $\eta_\ell(k) = 0$, then $S_\ell(k) = 0$ and also $f_\ell(k) = \frac{S_\ell(k)-1}{2ik} = \frac{i}{2k}$. Insert this into the formulas for the elastic and total cross sections:

$$\begin{aligned}\sigma_{\rm el} &= 4\pi\sum_{\ell=0}^{L}(2\ell+1)|f_\ell(k)|^2 = 4\pi\sum_{\ell=0}^{L}(2\ell+1)\frac{1}{4k^2} = \frac{\pi}{k^2}L^2 = \pi R^2 \; , \\ \sigma_{\rm tot} &= \frac{4\pi}{k}\mathrm{Im}\Big[f(\theta=0)\Big] = \frac{4\pi}{k}\sum_{\ell=0}^{L}(2\ell+1)\mathrm{Im}\big[f_\ell(k)\big] = \frac{4\pi}{k}\sum_{\ell=0}^{L}(2\ell+1)\frac{1}{2k} = 2\pi R^2 \; .\end{aligned} \tag{26}$$

The total cross section is twice as large as the elastic cross section! The difference between these two comprises the *inelastic* cross section. In fact, we would naively expect to have only such an inelastic cross section. What is happening here is that the sphere casts a shadow of size $\pi R^2$. Far away from the scatterer this shadow get filled in due to elastic scattering at its edge with very small angles. This contributes another $\pi R^2$, yielding a total cross section of $2\pi R^2$.

# 7. Single Atom Contacts

**(7.1)** [*6 points*] Consider first a conductor with a single conductance channel. We have $G = G_0 T_1$, with $G_0 = \frac{2e^2}{\hbar}$. This fixes the noise power completely, such that $S = S_0 F$, with $S_0 = 2eI$ and $F = 1 - T_1$. There is only one solution here.

For two channels ($N = 2$) we have $G = G_0 T$ with $T = T_1 + T_2$ and $S = S_0 F$, with

$$F = \frac{T_1(1-T_1) + T_2(1-T_2)}{T_1 + T_2}$$

Writing $T_2 = T - T_1$ in order to eliminate one of the two transmission numbers, we can express $F$ in terms of $T$ (which is given and fixed by the measured value of $G$), and $T_1$:

$$F = \frac{T_1(1-T_1) + (T-T_1)(1-T-T_1)}{T} = \frac{T - T^2 + 2TT_1 - 2T_1^2}{T}$$

The maximum of this expression is found by setting its derivative with respect to $T_1$ equal to zero,

$$\frac{\partial F}{\partial T_1} = 2 - 4\frac{T_1}{T} = 0$$

We find that the solution is $T_1 = T_2 = \frac{1}{2}T$. The maximum noise power is given by $S_{max} = S_0(1 - \frac{1}{2}T)$.

For arbitrary numbers of channels we may now guess that the answer is that all channels have equal transmission probabilities, $T_N = \frac{1}{N}T$. We can find this solution $\{T_1, T_2, ..., T_N\}$ by starting from the observation that the sum is fixed by

$$\sum_{n=1}^{N} T_n = T, \text{ with } T = \frac{G}{G_0}$$

This equation describes a plane in the $N$-dimensional space spanned by $\{T_1, T_2, ..., T_N\}$ perpendicular to the vector $(1, 1, ..., 1)$, constrained by the requirement $0 \leq T_n \leq 1$. In figure 1 we illustrate the possible solutions for the example of $N = 3$, and a total transmission $T$ larger than 1 and smaller than 3.

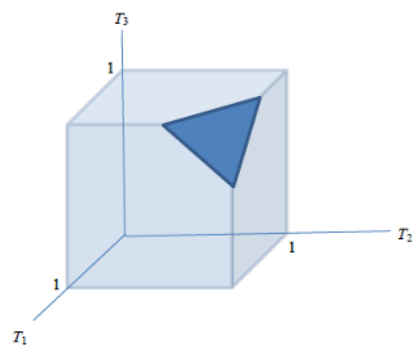



Figure 1

The problem now translates to finding which of these point has the maximum value of the noise power $S$, or for which

$$\sum_{n=1}^{N} T_n(1-T_n) = \sum_{n=1}^{N} T_n - \sum_{n=1}^{N} T_n^2$$

is maximal. Since the first term on the right is fixed by the total transmission $T$, we need to find the smallest value for the sum of squares of transmission values on the right. The sum of squares measures the square of the distance of the solution to the origin. In other words, the solution is the point at smallest distance to the origin. Obviously, this is the point at the center of the triangle in the figure above. The same analysis is valid for any dimension$N \geq 2$. The solution is given by taking all transmission values equal, $T_n = \frac{1}{N}T$. The maximum noise power is $S_{max} = S_0(1 - \frac{1}{N}T)$.

**(7.2)** [*6 points*] The minimum noise is found by similar reasoning. For $N = 1$ only a single unique solution exists, so that it is both minimum and maximum.

For $N \geq 2$ we follow the same reasoning as in 1. The problem now translates to finding which of the points on the plane spanned by the equation

$$\sum_{n=1}^{N} T_n = T$$

has the *minimum* value of the noise power $S$, or for which

$$\sum_{n=1}^{N} T_n(1 - T_n) = \sum_{n=1}^{N} T_n - \sum_{n=1}^{N} T_n^2$$

is minimal. Since the first term on the right is fixed by the total transmission T, we need to find the *largest* value for the sum of squares of transmission values on the right. The sum of squares measures the square of the distance of the solution to the origin. In other words, the solution is the point at *largest* distance to the origin. For $N = 3$ shown in the figure there are three solutions, given by the corner points of the triangle. These points are characterized by the property that all $T_n$ are equal to 1, except one. More generally (e.g. if we consider a smaller value $T < 2$) the corners of the plane section are on the edges of the cube, which means that all $T_n$ are either 1 or 0, expect one.

The same analysis is valid for any dimension $N \geq 2$. The solution is given by taking all transmission values equal to 1 or 0, expect for a single value for which $0 < T_n < 1$. The minimum noise power is given by just the contribution of this one channel,

$$S_{min} = S_0 \frac{T_n(1 - T_n)}{T}$$

# 8. Solar Sail

**(8.1)** [*2 points*] The force exerted by the photon pressure is equal to $F = 2IA/c$, as each photon of energy $\hbar\omega$ transfers $2\hbar k$ momentum upon reflection. The initial acceleration $a_0 = F_0/m = 2I_0A/mc$ follows from substitution of $I_0 = 1361$ W/m$^2$, $A = 6.4 \times 10^5$ m$^2$ and $m = 3$ kg, which yields $a_0 = 1.96$ m/s$^2$. The acceleration associated with gravity and the centrifugal force is $a_c = 4\pi r/T^2$, where $T \approx 365$ days $= 3.15 \times 10^7$ s, which yields $a_c = 0.0066$ m/s$^2$ and is thus a factor 300 smaller.

**(8.2)** [*1 point*] When we assume, as a first rough approximation, this acceleration to be constant, the distance travelled is simply given by $\Delta r = \frac{1}{2}at^2$. Substitution of $a_0 = 1.96$ m/s$^2$ and $\Delta r = 0.66r_0$ with $r_0 = 1.50 \times 10^8$ km yields $t = 3.18 \times 10^5$ s, which is only 3.68 days!

**(8.3)** [*2 points*] The distance dependence of the acceleration follows that of solar intensity and can be written $a = a_0(\frac{r_0}{r})^2$. The evolution equation for the distance $r(t)$ contains three forces: the radiation force, the gravitational pull of the sun, and the centrifugal force associated with the rotation (or the use of a rotating coordinate system). It can be written as

$$m\ddot{r} = ma_0\left(\frac{r_0}{r}\right)^2 - \frac{GMm}{r^2} + \frac{L}{mr^3},$$

where $L$ is the angular momentum, or as

$$\ddot{r} = a_0\left(\frac{r_0}{r}\right)^2 + a_c\left[\left(\frac{r_0}{r}\right)^3 - \left(\frac{r_0}{r}\right)^2\right] = (a_0 - a_c)\left(\frac{r_0}{r}\right)^2 + a_c\left(\frac{r_0}{r}\right)^3$$

**(8.4)** [*2 points*] From a mathematical perspective, the solar sail oriented towards the sun only modifies the relative strength of the quadratic term with respect to the cubic term in the evolution, as if the sail reduces the gravitational pull of the sun. The shape of the trajectory is therefore an ellipse, a parabola or a hyperbola. The case $a_0 \gg a_c$ corresponds to a hyperbola.

**(8.5)** [*2 points*] When the solar sail is 1000× heavier, and $a_0 \approx 0.30a_c$ the trajectory of the solar sail will be an ellipse and the maximum distance from the sun is not enough to reach Mars. You might have expected this, as the optimistic (upper bound of the) travel time calculated in 2 is now more than 116 days, which is long enough for the rotation around the sun to play an important role.

**(8.6)** [*3 points*] The total (=potential + kinetic) energy of the solar sail increases as a result of the work delivered by the photon force. As the deliverd power(=work per unit time) $P = \vec{F} \cdot \vec{v}$ increases with the angle between force and velocity, the energy transfer is much larger if we orient the solar sail such that the space craft is also propelled along its orbital velocity. The transfer of orbital momentum to space craft is optimum when the surface normal of the solar sail is oriented at 45° away from the sun. The optimum outwards trajectory requires more subtle tuning. When the solar sail is oriented such that its orbital momentum increases, it will enable a space craft of any mass to escape from the solar system, although this might take very long if the space craft is heavy.

# 9. The Quantum Mechanical Beamsplitter

**(9.1)** [*1 point*] Conservation of photons / Conservation of probability. Every photon has to go somewhere.

**(9.2)** [*2 points*] If this were not the case, it would be impossible to construct a unitary transformation. Said in a more physical way: if the number of inputs is smaller than the number of outputs, some photons will disappear when you reverse the direction of rays.

Theta tunes the splitting ratio of the beam splitter (from fully transmittive at 0 to fully reflecting at $\pi/2$).

**(9.3)** [*2 points*] $|1,0\rangle = a_1^\dagger|0,0\rangle = \cos(\theta)a_1'^\dagger + i\sin(\theta)a_2'^\dagger|0,0\rangle = \cos(\theta)|1,0\rangle + i\sin(\theta)|0,1\rangle$. The probability to find the photon in either arm is $\cos^2\theta$ or $\sin^2\theta$, respectively.

**(9.4)** [*2 points*] $|1,0\rangle = a_2^\dagger a_1^\dagger|0,0\rangle = (i\sin(\theta)a_1'^\dagger + \cos(\theta)a_2'^\dagger)(\cos(\theta)a_1'^\dagger + i\sin(\theta)a_2'^\dagger)|0,0\rangle = (i\cos\theta\sin\theta a_1'^\dagger a_1'^\dagger - \sin^2\theta a_1'^\dagger a_2'^\dagger + \cos^2\theta a_1'^\dagger a_2'^\dagger + i\cos\theta\sin\theta a_2'^\dagger a_2'^\dagger)|0,0\rangle$.

**(9.5)** [*2 points*] When $\theta = \pi/4$, the term with one photon in each arm vanishes. This means that the second photon has influenced where the first photon goes. In the light of this experiment, the statement that each photon interferes only with itself is no longer tenable.

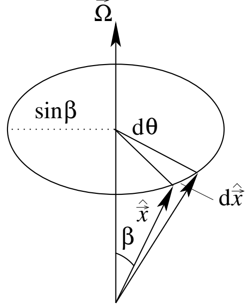
**(9.6)** [*3 points*] In this case, there is no interference, since one gets $|1,1\rangle \to (i\cos\theta\sin\theta a_{1,red}'^\dagger a_{1,red}'^\dagger - \sin^2\theta a_{1,red}'^\dagger a_{2,blue}'^\dagger + \cos^2\theta a_{1,blue}'^\dagger a_{2,red}'^\dagger + i\cos\theta\sin\theta a_{2,blue}'^\dagger a_{2,blue}'^\dagger)|0,0\rangle$. The two middle terms do not cross out. Each photon passes throught the beam splitter without sensing the other one.

# 10. Wind Drift of Icebergs Explained

**(10.1)** [*4 points*] We will consider a position vector $\vec{r}$ in the inertial frame (subscript I) and in the co-rotating frame (subscript R; unit vectors $\hat{\vec{x}}$, $\hat{\vec{y}}$, $\hat{\vec{z}}$). The rate of change of $\vec{r}$ in the inertial frame is

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_\mathrm{I} = \frac{\mathrm{d}}{\mathrm{d}t}(r_x\hat{\vec{x}} + r_y\hat{\vec{y}} + r_z\hat{\vec{z}} = \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_\mathrm{R} + r_x\frac{\mathrm{d}\hat{\vec{x}}}{\mathrm{d}t} + r_y\frac{\mathrm{d}\hat{\vec{y}}}{\mathrm{d}t} + r_z\frac{\mathrm{d}\hat{\vec{z}}}{\mathrm{d}t}\,.$$

The co-rotating unit vectors trace cones around $\vec{\Omega}$ in the inertial frame.




In a time $\mathrm{d}t$ the magnitude of the change in $\hat{\vec{x}}$ is $|\mathrm{d}\hat{\vec{x}}| = \sin\beta\mathrm{d}\theta$, where $\beta$ is the angle between $\vec{\Omega}$ and $\hat{\vec{x}}$. The direction of $\mathrm{d}\hat{\vec{x}}$ is perpendicular to both $\vec{\Omega}$ and $\hat{\vec{x}}$. Using that $\mathrm{d}\theta/\mathrm{d}t = |\vec{\Omega}|$ we find that

$$\frac{\mathrm{d}\hat{\vec{x}}}{\mathrm{d}t} = \vec{\Omega}\times\hat{\vec{x}}\,,$$

and thus that

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_\mathrm{I} = \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_\mathrm{R} + \vec{\Omega}\times\vec{r}\,.$$

We need to apply this relation twice to consider accelerations in the co-rotating frame:

$$\begin{aligned}\left(\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\right)_\mathrm{I} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_\mathrm{R} + \vec{\Omega}\times\vec{r}\right]_\mathrm{R} + \vec{\Omega}\times\left[\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_\mathrm{R} + \vec{\Omega}\times\vec{r}\right]_\mathrm{R}\\ &= \left(\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}\right)_\mathrm{R} + 2\vec{\Omega}\times\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_\mathrm{R} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})\,,\end{aligned}$$

where we can recognise the linear acceleration in the co-rotating frame, the Coriolis acceleration and the centrifugal acceleration, respectively. The centrifugal term can be rewritten as

$$\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) = \vec{\Omega}\times(\vec{\Omega}\times\vec{r}_\perp) = -\Omega^2\vec{r}_\perp\,,$$

where $\vec{r}_\perp$ is the projection of $\vec{r}$ on the plane perpendicular to $\vec{\Omega}$; again $\Omega = |\vec{\Omega}|$. Furthermore, this term can be written as the gradient of a scalar as

$$-\Omega^2\vec{r}_\perp = -\vec{\nabla}(\tfrac{1}{2}\Omega^2 r_\perp^2)\,,$$

with $r_\perp = |\vec{r}_\perp|$.

Applying this relation to the Navier–Stokes equation, we arrive at

$$\frac{\partial\vec{v}}{\partial t} + 2\vec{\Omega}\times\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}(p - \tfrac{1}{2}\rho\Omega^2 r_\perp^2) + \nu\nabla^2\vec{v}\,,$$

where we have used that spatial derivatives do not change, only derivatives to time. We can introduce the reduced pressure $P = p - \frac{1}{2}\rho\Omega^2 r_\perp^2$ to fully account for centrifugal acceleration.

**(10.2)** [*3 points*] The geostrophic balance reveals that $\partial P/\partial z = 0$. Taking the derivative to $z$ of equation the equation

$$2\Omega\hat{\vec{z}}\times\vec{v} = -\frac{1}{\rho}\vec{\nabla}P$$

and switching the order of the derivatives of the pressure terms reveals that

$$\frac{\partial \vec{v}}{\partial z} = \vec{0}\,.$$

Apparently, under the geostrophic balance all velocity components are independent of height. This must change close to the surface, where viscosity comes into play.

**(10.3)** [*5 points*] The boundary-layer equations for the horizontal velocity components are

$$-2\Omega v_y = \nu \nabla^2 v_x\,,$$

$$2\Omega v_x = \nu \nabla^2 v_y\,.$$

Introducing the complex velocity $\phi = v_x + \mathrm{i}v_y$ we can rewrite this into a single equation:

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{2\Omega \mathrm{i}}{\nu}\phi\,.$$

This equation has a general solution

$$\phi = A\exp\left(\frac{(1+\mathrm{i})z}{\sqrt{\nu/\Omega}}\right) + B\exp\left(\frac{-(1+\mathrm{i})z}{\sqrt{\nu/\Omega}}\right)\,,$$

where $A$ and $B$ are constants to be determined with the boundary conditions. We can see that the typical thickness of the boundary layer is $\delta = \sqrt{\nu/\Omega}$. Applying the boundary conditions, we find that $B$ must be zero for the solution to disappear for $z \to -\infty$. At $z = 0$ we find that

$$A = \frac{\tau\delta(1-\mathrm{i})}{2\rho\nu}\,.$$

Thus the solution is found to be

$$v_x = \frac{\tau}{\rho\sqrt{2\Omega\nu}}\exp\left(\frac{z}{\delta}\right)\cos\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)\,,$$

$$v_y = -\frac{\tau}{\rho\sqrt{2\Omega\nu}}\exp\left(\frac{z}{\delta}\right)\sin\left(-\frac{z}{\delta} + \frac{\pi}{4}\right)\,.$$

We can see that the flow direction and magnitude is changing as a function of $z$. At $z = 0$ we find that

$$(v_x, v_y) = \frac{\tau}{2\rho\sqrt{\Omega\nu}}(1,1)\,.$$

So the theoretical drift direction is 45° to the right of the wind. Given the rigorous approximations made in this derivation and the multitude of effects *not* taken into account (waves, density stratification, icebergs can be large and thus cover a rather wide range of $z$, etc) this is a remarkably appropriate result!

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