You guessed it: phase transitions in a 2D lattice gas

Consider a two-dimensional (2D) square lattice consisting of N sites. Each site can be either occupied ($n_i = 1$) or empty ($n_i = 0$). Particles in neighboring sites interact attractively. We will model this attraction as a constant negative energy $-\epsilon$. Moreover, a filled site contributes an energy $-\mu$ to the total energy of the system. Do not make assumptions yet on the sign of μ. The hamiltonian of the system is given by

$$H = -\epsilon \sum_{(i,j)} n_i n_j - \mu \sum_i n_i$$

where the first sum runs over nearest neighbor pairs. In what follows we will call the average number of particles per site ($\langle n_i \rangle$) the density of the system. In what follows, consider the lattice is canonical conditions.

(a) Discuss qualitatively (i.e. without necessarily writing any equation) the behavior of the density at high temperature

Solution

At high temperatures the interaction energy between neighboring particles is small compared to the thermal energy. The same is holds for the energy of occupying a site. Hence, the probability of a site being occupied should be equal to the probability of a site remaining empty. This means the density takes the value 1/2 at high temperatures, which corresponds to a gas-like behavior. .... 2 p

(b) Discuss qualitatively the behavior of the density at low temperature as function μ

Solution

At low temperatures the discussion is more complex, as it depends the energy μ. If $\mu \ll \epsilon$ the hamiltonian will be minimized if all sites are filled, hence $\rho = 1$, i.e. a liquid-like behavior. On the other hand, if μ is small (or negative, even), the hamiltonian will be minimized if no particles are present, so $\rho = 0. \cdots \ldots \ldots \cdot 2$ p

(c) Assuming that the fluctuations in density are very small and that the system is translationally invariant (say, by a distance equal to the lattice constant), derive a mean field expression for the hamiltonian and show that it is equal to

$$H_{\mathrm{MF}} = -4\epsilon\rho \sum_{i=1}^{N} n_i + 2\epsilon\rho^2 N - \mu \sum_i n_i$$

or, equivalently for a single site

$$h_{\mathrm{MF}} = -4\epsilon\rho n_i + 2\epsilon\rho^2 n_i - \mu n_i$$

where $\rho \equiv \langle n_i \rangle$ for simplicity. It might help here to write $n_i = \rho + \delta n_i$

Solution

We write the number of particles in site i as

$$n_i = \rho + \delta n_i$$

the interaction term contains the product $n_i n_j$ which expands to

$$n_i n_j = (\rho + \delta n_i)(\rho + \delta n_j) = \rho^2 + \rho\delta n_i + \rho\delta n_j + \delta n_i \delta n_j$$

Since fluctuations in density are small we neglect $\delta n_i \delta n_j$. Adding and subtracting ρ^2 from the above, we obtain

$$n_i n_j = \rho^2 + \rho\delta n_i + \rho^2 + \delta n_j - \rho^2 = n_i + n_j - \rho^2 \ldots\ldots\ldots\ldots\ldots\cdots 1\mathrm{p}$$

The sum over neighboring sites can be written as

$$\sum_{(i,j)} = \frac{1}{2}\sum_i \sum_{j\in \mathrm{mn}(i)} = 2\sum_i$$

where nn(i) means "nearest neighbors of cell i ". There are 4 of those. Finally, we observe that $\sum_i n_i = \sum_j n_j$. Putting it all together we have

$$\begin{aligned} H_{\mathrm{MF}} &= -2\epsilon\sum_i n_i - 2\epsilon\sum_j n_j + 2\epsilon\sum_i \rho^2 - \mu\sum_i n_i \\ &= -4\epsilon\rho\sum_i n_i + 2\epsilon\rho^2 N - \mu\sum_i n_i \\ &= \sum_i -4\epsilon\rho n_i + 2\epsilon\rho^2 - \mu n_i \cdots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

(d) Compute the large temperature limit of the density ρ

Solution

The single-site partition function in the mean field approximation is given by

$$z = \sum_{n_i=0,1} e^{-\beta h(n_i)} = 1 + e^{-\beta\left(-4\epsilon\rho + 2\epsilon\rho^2 - \mu\right)}$$

with $\beta = 1/k_B T$. The average number of particles in a site is given by

$$\langle n_i \rangle = \frac{\sum_{n_i=0,1} n_i e^{-\beta h(n_i)}}{z} = \frac{e^{-\beta\left(-4\epsilon\rho+2\epsilon\rho^2-\mu\right)}}{1+e^{-\beta(-4\epsilon\rho+2\epsilon\rho^2-\mu)}} \cdots\cdots\cdots\cdots 0.5\text{p}$$

At high temperatures, $\beta \to 0$ so $\langle n_i \rangle \equiv \rho = 1/2$

(e) In certain conditions, the high temperature behavior of ρ from the previous point will be reached at a finite temperature T_c. This is called a continuous phase transition. Find the value of μ for which the system has this property. Find the critical temperature T_c. Hint: the order parameter here is the density ρ.

Solution

We rewrite the density from the previous point as

$$\rho = \frac{1}{1+e^{\beta(-4\epsilon\rho+2\epsilon\rho^2-\mu)}}$$

The condition that ρ attains the high-temperature limit at a finite temperature (call it critical) implies that

$$-4\epsilon\rho\,(T=T_c) + 2\epsilon\rho^2\,(T=T_c) - \mu = 0$$

By definition of the critical temperature we have $\rho\,(T=T_c) = 1/2$ so

$$\mu = -2\epsilon + \frac{1}{2}\epsilon = -\frac{3}{2}\epsilon \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 1p$$

To find the critical temperature, we need to solve the self-consistent equation above. This can be achieved by expanding it near the critical value. To simplify calculations we write

$$\rho(\beta) = \frac{1}{1+e^{f(\beta)}}$$

where $f(\beta) = \beta\left(-4\epsilon\rho + 2\epsilon\rho^2 - \mu\right)$. Near the critical temperature, we have

$$\rho \simeq \frac{1}{2} + \left.\left(\frac{\partial\rho}{\partial T}\right)\right|_{T=T_c}(T-T_c) = \frac{1}{2} - \frac{1}{k_b T_c^2}\left.\left(\frac{\partial\rho}{\partial\beta}\right)\right|_{\beta=\beta_c}(T-T_c)$$

The partial derivative is given by

$$\left(\frac{\partial\rho}{\partial\beta}\right) = -\frac{e^{f(\beta)}}{\left(1+e^{f(\beta)}\right)^2} f'(\beta)$$

The fraction is 1 at the critical temperature on account of $f\,(\beta_c) = 0$ as discussed above. It remains to evaluate $f'(\beta)$ at the critical temperature

$$f'\left(\beta_c\right) = -4\epsilon\rho\left(\beta_c\right) + 2\epsilon\rho^2\left(\beta_c\right) - \mu - 4\beta_c\epsilon\left(\frac{\partial\rho}{\partial\beta}\right)_{\beta=\beta_c} + 4\beta_c\epsilon\rho\left(\beta_c\right)\left(\frac{\partial\rho}{\partial\beta}\right)_{\beta=\beta_c}$$

The first three terms add up to 0 and we have

$$\left(\frac{\partial\rho}{\partial\beta}\right)_{\beta=\beta_c} = 4\beta_c\epsilon\left(\frac{\partial\rho}{\partial\beta}\right)_{\beta=\beta_c} - 2\beta_c\left(\frac{\partial\rho}{\partial\beta}\right)_{\beta=\beta_c}$$

which can be rearranged to

$$\left(\frac{\partial\rho}{\partial\beta}\right)_{\beta=\beta_c}\left(1-2\beta_c\epsilon\right) = 0$$

giving

$$\beta_c = \frac{1}{2\epsilon} \to T_c = \frac{2\epsilon}{k_b} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot 1.5\text{p}$$

The angular momentum stored in a combination of steady electric and magnetic fields

1p din oficiu

a) Consider the vertical z-axis pointing upwards. Since the superconducting tube is suposed to be infinitely long, we set $\mathbf{B}\left(r'\right) = B\mathbf{k}$ for $r' < r$ (i.e. a counter-clockwise supercurrent), $\mathbf{B}\left(r'\right) = \mathbf{0}$ for $r' > r$, then $\Phi = \pi r^2 B$.

We have

$$\oint_{C(R_1)} \mathbf{A}_1 \cdot d\mathbf{r}_1 = \Phi_1 = \pi R_1^2 B = \Phi \frac{R_1^2}{r^2} \tag{1}$$

and

$$\oint_{C(R_2)} \mathbf{A}_2 \cdot d\mathbf{r}_2 = \Phi_2 = \Phi \tag{2}$$

where $C\left(R_1\right)$ and $C\left(R_2\right)$ stand for the circles of radii R_1 and R_2, respectively, and $\mathbf{A}_1, \mathbf{A}_2$ are magnetic potential vectors both tangential to the circles $C\left(R_1\right), C\left(R_2\right)$ in the direction of supercurrent. Accounting for the rotational symmetry, A_1, A_2 are constants, and we get the magnetic vector potential in the regions of the two charge distributions

$$\mathbf{A}_1 = \Phi \frac{\mathbf{k} \times \mathbf{R}_1}{2\pi r^2} \tag{3}$$

and

$$\mathbf{A}_2 = \Phi \frac{\mathbf{k} \times \mathbf{R}_2}{2\pi R_2^2}. \tag{4}$$

Since the superficial charge distributions are uniform and their initial state has zero mechanical momenta, the infinitesimal field momenta of the charge elements $dq_1 = \sigma dS_1$ and $dq_2 = -\sigma dS_2$ in the two distributions are given by

$$d\mathbf{P}_1 = \mathbf{A}_1 dq_1 \text{ and } d\mathbf{P}_2 = \mathbf{A}_2 dq_2. \tag{5}$$

The corresponding field angular momenta read

$$\mathbf{L}_1 = \int_{D_1} \mathbf{R}_1 \times d\mathbf{P}_1 = \frac{\Phi\sigma}{2\pi r^2} \int_{D_1} \mathbf{R}_1 \times \left(\mathbf{k} \times \mathbf{R}_1\right) dS_1 = \frac{\Phi\sigma R_1^2 S_1}{2\pi r^2}\mathbf{k} = \frac{\Phi\sigma R_1^3 L}{r^2}\mathbf{k} \tag{6}$$

and

$$\mathbf{L}_2 = \int_{D_1} \mathbf{R}_2 \times d\mathbf{P}_2 = -\frac{\Phi\sigma}{2\pi R_2^2} \int_{D_1} \mathbf{R}_2 \times \left(\mathbf{k} \times \mathbf{R}_2\right) dS_2 = -\Phi\sigma R_2 L\mathbf{k}. \tag{7}$$

In the above eqns. D_1, D_2 stand for the cylindrical sheets of the two charge distributions, and dS_1, dS_2 are infinitesimal elements of areas.

Therefore, the angular momentum $\mathbf{L}$ stored in the field is

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = \Phi \sigma L R_2 \left(\frac{R_1^3}{R_2 r^2} - 1 \right) \mathbf{k}. \tag{8}$$

Note that $\mathbf{L} \neq \mathbf{0}$, antiparallel to $\mathbf{B}$ if $\sigma > 0$, and parallel to $\mathbf{B}$ if $\sigma < 0$.
b) Without supercurrent there is no magnetic flux and, according to (8), no angular momentum will be stored in the field. As a consequence, if the charge distributions do not rotate, the mechanical angular momenta will also be zero and the law of conservation of angular momentum we be violated.

According to Faraday's electromagnetic induction law and Lenz rule, the induced electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ at the location of charge distributions will try to keep the magnetic flux in place. We can write

$$-\frac{d\Phi_1}{dt} = \frac{R_1^2}{r^2} \left| \frac{d\Phi}{dt} \right| = \oint_{c(R_1)} \mathbf{E}_1 \cdot d\mathbf{r}_1, \tag{9}$$

and

$$-\frac{d\Phi_2}{dt} = \left| \frac{d\Phi}{dt} \right| = \oint_{c(R_1)} \mathbf{E}_2 \cdot d\mathbf{r}_2 \tag{10}$$

with the requirement that both induced electric fields $\mathbf{E}_1, \mathbf{E}_2$ be tangential to $C\left(R_1\right), C\left(R_2\right)$ and in the direction of initial supercurrent.

Therefore the electric force $\mathbf{F}_1$ acting on σ charge distribution will rotate it in the direction of supercurrent if $\sigma > 0$, and in oposite direction if $\sigma < 0$. Similar considerations apply to rotation of $-\sigma$ charge distribution.
c) From eqs. (9) and (10), we get

$$\frac{R_1}{2\pi r^2} \left| \frac{d\Phi}{dt} \right| = E_1 \quad \text{and} \quad \frac{1}{2\pi R_2} \left| \frac{d\Phi}{dt} \right| = E_2$$

or,

$$\mathbf{E}_1 = \frac{\mathbf{k} \times \mathbf{R}_1}{2\pi r^2} \left| \frac{d\Phi}{dt} \right| \text{ and } \mathbf{E}_2 = \frac{\mathbf{k} \times \mathbf{R}_2}{2\pi R_2^2} \left| \frac{d\Phi}{dt} \right| \tag{12}$$

The electric forces on the charge distributions are

$$\mathbf{F}_1 = q_1 \mathbf{E}_1 = \left(\mathbf{k} \times \mathbf{R}_1 \right) \frac{R_1 L \sigma}{r^2} \left| \frac{d\Phi}{dt} \right| \text{ and } \mathbf{F}_2 = q_2 \mathbf{E}_2 = -\frac{\sigma L}{R_2} \left| \frac{d\Phi}{dt} \right| \left(\mathbf{k} \times \mathbf{R}_2 \right), \tag{13}$$

providing the mechanical angular momenta contributions

$$d\mathbf{L}_1' = \mathbf{R}_1 \times \mathbf{F}_1 dt = \mathbf{R}_1 \times \left(\mathbf{k} \times \mathbf{R}_1 \right) \frac{R_1 L \sigma}{r^2} |d\Phi|, d\mathbf{L}_2' = \mathbf{R}_2 \times \mathbf{F}_2 dt = -\mathbf{R}_2 \times \left(\mathbf{k} \times \mathbf{R}_2 \right) \frac{\sigma L}{R_2} |d\Phi|, \tag{14}$$

or,

$$\mathbf{L}'_1 = \frac{\Phi\sigma R_1^3 L}{r^2}\mathbf{k} \text{ and } \mathbf{L}'_2 = -\Phi\sigma R_2 L\mathbf{k} \tag{15}$$

These mechanical angular momenta are the same as those given by eqs. (6), (7).

As an important conclusion, after dropping the supercurrent the mechanical angular momentum ensures the conservation of the initial angular momentum, as expected.

$$\mathbf{L}' = \mathbf{L}'_1 + \mathbf{L}'_2 = \mathbf{L} \tag{16}$$

Michelson interferometry

I. The optical path difference in the interference pattern is:
$\delta = 2t\cos i \pm \lambda/2;$ 0.25p
and in the centre of the interference pattern:
$\delta = 2t \pm \lambda/2.$
0.25 p

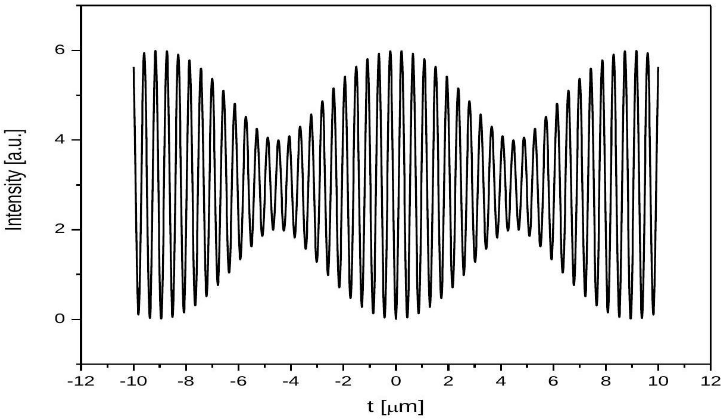
For a monochromatic source the intensity distribution in the interferogram is:
$I = 2I_0\left(1+\cos\frac{2\pi\delta}{\lambda}\right) = 2I_0\left(1-\cos\frac{4\pi t}{\lambda}\right)$
0.5 p

For a two monochromatic source the intensity distribution in the interferogram is:
For a monochromatic source the intensity distribution in the interferogram is:
$I = 2I_{01}\left(1-\cos\frac{4\pi t}{\lambda_1}\right) + 2I_{02}\left(1-\cos\frac{4\pi t}{\lambda_2}\right)$
0.5 p

If $\lambda_{1.2} = \bar{\lambda} \pm \Delta\lambda/2, \Delta\lambda \ll \lambda_{1,2}$:
$I = 2I_{01}\left[1-\cos\frac{4\pi t}{\bar{\lambda}}\left(1-\frac{\Delta\lambda}{2\bar{\lambda}}\right)\right] + 2I_{02}\left[1-\cos\frac{4\pi t}{\bar{\lambda}}\left(1+\frac{\Delta\lambda}{2\bar{\lambda}}\right)\right]$ 0.5p

$$I = 2\left(I_{01}+I_{02}\right) - 2\left(I_{01}+I_{02}\right)\cos\frac{4\pi t}{\bar{\lambda}}\cos\frac{2\pi\Delta\lambda t}{\bar{\lambda}^2} + 2\left(I_{01}-I_{02}\right)\sin\frac{4\pi t}{\bar{\lambda}}\sin\frac{2\pi\Delta\lambda t}{\bar{\lambda}^2}$$

Where $2\left(I_{01}+I_{02}\right) = 3, 2\left(I_{01}-I_{02}\right) = 1,\ \mathrm{K}_1 = 4\pi/\bar{\lambda}, \mathrm{K}_2 = 2\pi\Delta\lambda/\bar{\lambda}^2.$
a. $I_{01}/I_{02} = 2;$ 0.5p
b. $\bar{\lambda} = 873.27$ nm;0.5
c. $\Delta\lambda = 42.24$ nm;0.5 p



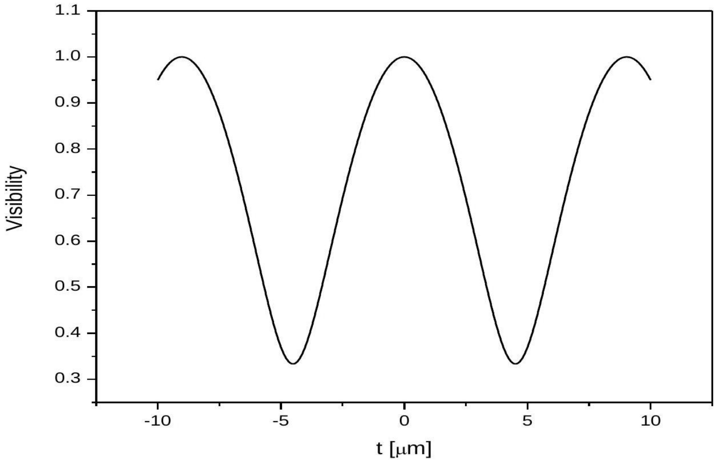
II. a. 1 p

b. $I_{\max} = 3 + \sqrt{(3\cos K_2 t)^2 + (\sin K_2 t)^2} = 3 + \sqrt{1 + 8\cos^2 K_2 t}$ 0.5p

$I_{\min} = 3 - \sqrt{(3\cos K_2 t)^2 + (\sin K_2 t)^2} = 3 - \sqrt{1 + 8\cos^2 K_2 t}$ 0.5p

$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{\sqrt{1+8\cos^2 K_2 t}}{3}$

0.5p



III. The envelope of the oscillating function is multiplied by a broad gaussian function. From the convolution theorem this implies the spectrum is convolved by a narrow gaussian function. This could arise from Doppler broadening of the atomic emission lines.

Heat Capacities

1. a) Show that

$$C_p - C_V = \left[p + \left(\frac{\partial U}{\partial V}\right)_T\right]\left(\frac{\partial V}{\partial T}\right)_p$$

where C_p and C_V are the heat capacities at constant pressure and volume, respectively. U and V are internal energy and volume.

b) Use the above equality and the relation

$$p + \left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V$$

to determine the expression of $(C_p - C_V)$ as a function of state variables (T, V, p), for a mole of a Van der Waals gas.

Solution

a) From the expression of enthalpy one obtains:

$$(1) \quad H = U + pV, \tag{1}$$

$$(2) \quad dH = dU + p\,dV + V\,dp = \delta Q + V\,dp. \tag{2}$$

From (2) one obtains:

$$(3) \quad \left(\frac{\partial H}{\partial T}\right)_p = \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p.$$

Definitions:

$$(4) \quad C_p = \left(\frac{\partial Q}{\partial T}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p, \quad (5) \quad C_V = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V.$$

State functions:

$$(6) \quad dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV, \tag{3}$$

$$(7) \quad dV = \left(\frac{\partial V}{\partial T}\right)_p dT + \left(\frac{\partial V}{\partial p}\right)_T dp. \tag{4}$$

Combining (6) and (7) and dividing by dT at constant p gives:

$$(9) \quad \left(\frac{\partial U}{\partial T}\right)_p = \left(\frac{\partial U}{\partial T}\right)_V + \left(\frac{\partial U}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p.$$

Replacing (9) in (3) yields:

$$(10) \quad \left(\frac{\partial H}{\partial T}\right)_p = \left(\frac{\partial U}{\partial T}\right)_V + [\left(\frac{\partial U}{\partial V}\right)_T + p]\left(\frac{\partial V}{\partial T}\right)_p.$$

Using (4) and (5) gives:

$$(11) \quad C_p - C_V = [\left(\frac{\partial U}{\partial V}\right)_T + p] \left(\frac{\partial V}{\partial T}\right)_p .$$

b) Applying the given relation transforms (11) to:

$$(12) \quad C_p - C_V = T \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p .$$

For a Van der Waals gas:

$$(13) \quad \left(\frac{\partial p}{\partial T}\right)_V = \frac{R}{V-b}, \quad (14) \quad \left(\frac{\partial V}{\partial T}\right)_p = \frac{R}{\frac{RT}{V-b} - \frac{2a(V-b)}{V^3}} .$$

Hence:

$$(15) \quad C_p - C_V = R \left[1 - \frac{2a\left(1-\frac{b}{V}\right)^2}{VRT}\right]^{-1} .$$

Score:

- Deriving relation (11) — 5 points
- Deriving relation (15) — 4 points
- Ex officio — 1 point

Experimentul Davis

Detectorul Davis folosit pentru studiul dezintegrării β şi punerii în evidenţă prin metode directe a existenţei neutrinului conţinea 600 tone de $C_2\mathrm{Cl}_4$. Clorul conţinea 24.47% din izotopul de ${}^{37}_{17}\mathrm{Cl}$. Secţiunea eficace pentru reacţia ${}^{37}_{17}\mathrm{Cl} + v_e \to e^- + {}^{37}_{18}\mathrm{Ar}$, mediată pe partea de peste pragul de producere a spectrului neutrinilor obţinuţi din dezintegrarea ${}^{8}_{5}\quad B \to {}^{8}_{4}\quad \mathrm{Be} + e^+ + v_e$ este de 10^{-46} m^2. Producerea de izotopi de ${}^{37}_{18}$ Areste de un izotop la 2 zile. Să se calculeze rata de captură şi fluxul neutrinilor detectabili.

Se dă masa moleculară a $C_2\mathrm{Cl}_4$, anume:165.83.

Barem pentru corectare

Pentru rezolvarea problemei este necesară determinarea numărului de atomi de clor, anume izotopul ${}^{37}_{17}\mathrm{Cl}$. Se vor lua în considerare masa moleculară a tetraclorurii de carbon, $C_2\mathrm{Cl}_4$, masa totală a detectorului, precum şi abundenţa izotopului ${}^{37}_{17}\mathrm{Cl}$; se poate scrie:

$$n({}^{37}_{17}\mathrm{Cl}) = \frac{m(\mathrm{C_2Cl_4})\, a_{\mathrm{iz}}({}^{37}_{17}\mathrm{Cl})\, n_{\mathrm{at}}({}^{37}_{17}\mathrm{Cl})}{M({}^{37}_{17}\mathrm{Cl})}\, N_A.$$

1,50 puncte Efectuând calculele, se obţine:

$$n({}^{37}_{17}\mathrm{Cl}) = \frac{6\times 10^5\,\mathrm{g}\times 0.2447\times 4}{165.83}\times 6.023\times 10^{26} = 2.13\times 10^{30}\ \text{atomi}.$$

0,50 puncte Determinarea ratei de captură medii se face luând în considerare datele problemei; se poate scrie, pentru un singur atom relaţia de mai jos:

$$\frac{\mathrm{dn}}{\mathrm{dt}} = \frac{1}{2\mathrm{zile}} = \frac{1}{2.86400\ \mathrm{s}} \simeq 5,79\times 10^{-6}\ \mathrm{s}^{-1}$$

1,50 puncte Dacă se ia în considerare numărul total de atomi de ${}^{37}_{17}\mathrm{Cl}$ se obţine:

$$\left(\frac{\mathrm{dn}}{\mathrm{dt}}\right)_{\mathrm{total}} = \frac{1}{n\,({}^{37}_{17}\mathrm{Cl})}\cdot\frac{\mathrm{dn}}{\mathrm{dt}}$$

1,50 puncte

Cu ajutorul acestei rate şi a secţiunii eficace se poate determina fluxul de neutrini detectabili, astfel:

$$\left(\frac{\mathrm{dn}}{\mathrm{dt}}\right)_{\mathrm{total}} = \Phi_v\cdot\sigma_v,$$

de unde rezultă:

$$\Phi_\nu = \frac{dn/dt}{n({}^{37}_{17}\mathrm{Cl})\,\sigma_\nu}.$$

Se obţine:

$$\Phi_\nu = \frac{5.79 \times 10^{-6}\,\mathrm{s}^{-1}}{2.13 \times 10^{30} \times 10^{-46}\,\mathrm{m}^2} \approx 2.72 \times 10^{10}\,\mathrm{neutrini}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}.$$

Se acordă $\mathbf{1,00}$ puncte din oficiu

Total = 10 puncte

‚Mezonul' μ

Descoperirea miuonului a introdus, la momentul iniţial, o uşoară confuzie cu cuanta de schimb a interacţiilor tari la nivel nuclear, datorită valorii masei de repaus. Ulterior, după descoperirea pionului, în anul 1947, de Powell şi colaboratorii, lucrurile s-au clarificat, miuonul devenind extrem de important pentru studiul interacţiei slabe şi fundamentarea modelul standard, dar şi pentru investigarea proceselor cosmologice fundamentale. Studiind dezintegrarea „mezonului" μ prin metoda coincidenţelor întârziate, în intervalul de timp $0 - 210^{-6}$ secunde au fost înregistrate 200 dezintegrări, iar în intervalul $0 - 610^{-6}$ secunde au fost înregistrate 310 dezintegrări. Presupunând că dezintegrarea respectă legea dezintegrării radioactive, să se calculeze timpul de viaţă pentru acest lepton.

Barem pentru corectare

Numerele de miuoni nedezintegraţi după cele două intervale de timp, în ipoteza măsurării simulane, pentru a avea acelaşi număr iniţial de leptoni, se calculează astfel :

$$N_1 = N_0\left(1 - e^{-\frac{t_1}{\tau}}\right)$$
$$N_2 = N_0\left(1 - e^{-\frac{t_2}{\tau}}\right)$$

3,00 puncte Pentru determinarea timpului de viaţă se poate face raportul între cele două relaţii anterioare, cu luarea în considerare a raportului dintre timpii de măsurare pentru cele două cazuri. Se obține relaaţia de mai jos:

$$\frac{N_2}{N_1} = \frac{1 - e^{-\frac{2t_1}{\tau}}}{1 - e^{-\frac{t_1}{\tau}}} = \frac{310}{200} = 1,55$$

2,00 puncte O cale de rezolvare a ecuaţiei de mai sus este următoarea: se face notatia $e^{-\frac{t_1}{\tau}} = x$ şi se introduce în expresie anterioară, obţinându-se următoarea ecuaţie de gradul al doilea:

$$x^2 - 1,55x + 0,55 = 0$$

Soluţiile ecuaţiei de mai sus sunt:

$$x_{1,2} = \frac{1,55 \pm \sqrt{1,55^2 - 4.0,55}}{2} = \frac{1,55 \pm 0,45}{2} = \{1\;|$$

2,00 puncte Soluţia $x_1 = 1$ nu are sens, deoarece timpul de viaţă ar trebui să fie infinit (particulă stabilă). 1,00 puncte Pentru $x = e^{-\frac{t_1}{\tau}} = 0,55$ se obţine, pentru timpul de viaţă mediu, următoarea valoare:

$$\tau \simeq 2,5 \cdot 10^{-6} \text{ secunde}$$

Se acordă 1,00 puncte din oficiu

Total = 10 puncte

Modele de structură nucleară

Spinul şi paritatea, I^π, precum şi energia E pentru starea fundamentală şi o secvenţă de stări excitate ale nucleului de $^{170}_{72}$ Hf sunt următoarele:

I^π	0^+	2^+	4^+	6^+	8^+
E[keV]	0	100	321	641	1041

Ce fel de niveluri de energie sunt?

Să se calculeze momentul de inerţie al nucleului în cel puţin 2 stări excitate. Să se compare valorile obținute cu cele calculate în ipoteza că nucleul este o sferă rigidă care se roteşte. Momentul de inerţie al sferei este $\Im_{\rm sf} = \frac{2}{5}\rm MR^2$. Parametrul $\rm r_0$ în expresia razei nucleare se va lua egal cu 1.3 Fm. Să se comenteze rezultatul. Se dă valoarea produsului dintre constanta redusă a lui Planck şi viteza luminii în vid, în sistemul natural de unităţi, anume: $\hbar c = 197,3\rm MeV.Fm$

Barem de corectură

Nivelurile de energie din tabelul dat în enunţ sunt nivelurile de rotaţie ale nucleului de $^{170}_{72}$ Hf. Acest lucru se deduce din valorile spinului, I, numere pare pentru toate, şi din valoarea constantă a parităţii nivelurilor de energie.

1,50 puncte Energia stărilor de rotaţie corespunzătoare se calculează astfel:

$$E^I_{\rm rot} = \frac{I(I+1)\hbar^2}{2\Im}.$$

Relaţia de mai sus se mai poate scrie sub forma:

$$E^I_{\rm rot} = \frac{I(I+1)\hbar^2 c^2}{2\Im c^2},$$

ceea ce permite folosirea valorii produsului dintre constanta redusă a lui Planck şi viteza luminii în vid, în sistemul natural de unităţi, anume: $\hbar c = 197,3\rm MeV.Fm$

1,00 puncte Pentru calcularea momentelor de inerţie asociate se foloseşte relaţia dată pentru o sferă rigidă: $\Im_{\rm sf} = \frac{2}{5}\rm MR^2$. Având în vedere ultima expresie a energiei de rotaţie, se poate scrie expresia momentului de inerţie astfel:

$$\Im c^2 = \frac{2}{5}\rm Mc^2 \mathit{R}^2$$

1,50 puncte Efectuând calculele pentru nucleul de ${}^{170}_{72}$ Hf rezultă:

$$\Im c^2 = \frac{2}{5}\mathrm{M}c^2 R^2 = \frac{2}{5} 170 \cdot 931,5\mathrm{MeV} \cdot \left(1,3 \cdot 170^{1/3}\right)^2 \mathrm{Fm}^2 =$$
$$63342\mathrm{MeV} \cdot 51,86\mathrm{Fm}^2 = 3,285 \times 10^6 \mathrm{MeV} \cdot \mathrm{Fm}^2$$

1,00 puncte Valorile momentelor de inerţie pentru două stări excitate, 2^+şi 8^+, de exemplu, se calculează cu ajutorul relaţie:

$$\Im^2 = \frac{I(I+1)\hbar^2 c^2}{2E^I_{\mathrm{rot}}}.$$

Valorile obţinute sunt: (a) pentru starea 2^+:

$$\Im^2 c^2 \left(2^+\right) = \frac{I(I+1)\hbar^2 c^2}{2E^I_{\mathrm{rot}}} = \frac{2 \cdot 3 \cdot 197,3^2}{2 \cdot 0,100} \mathrm{MeV}^2 \mathrm{Fm}^2 \simeq 1,168 \times 10^6 \mathrm{MeV.Fm}^2$$

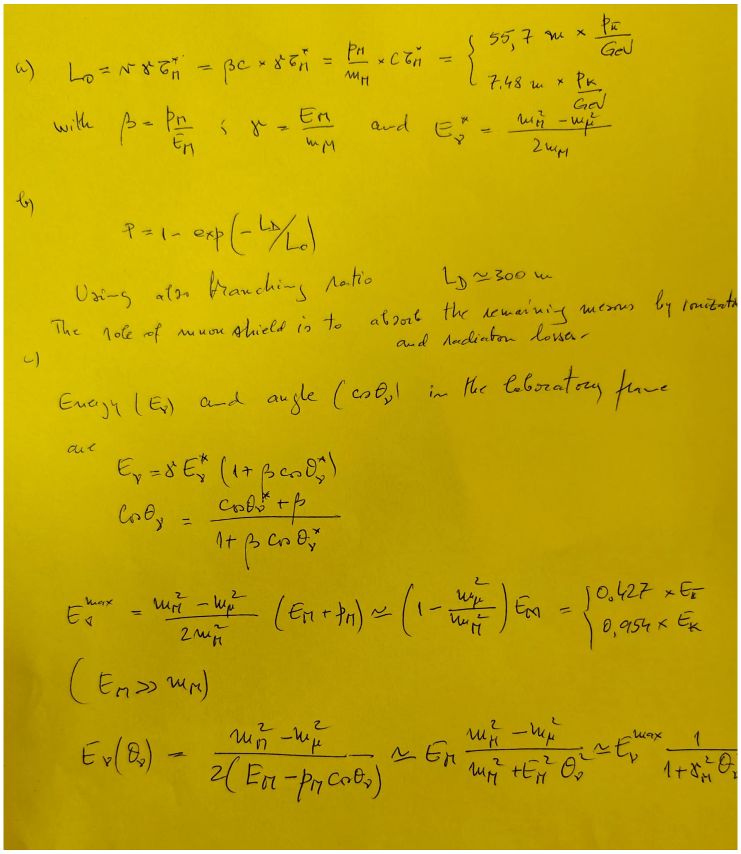
0,75 puncte (b) pentru starea 8^+:

$$\Im^2 \left(8^+\right) = \frac{I(I+1)\hbar^2 c^2}{2E^I_{\mathrm{rot}}} = \frac{8.9 \cdot 197,3^2}{2 \cdot 1,041} \mathrm{MeV.Fm}^2 \simeq 1,346 \times 10^6 \mathrm{MeV.Fm}^2 \qquad \text{0,75 puncte}$$

Se observă că valorile determinate din valorile energiilor de rotaţie sunt de 2-3 ori mai mici decât cele determinate folosind expresia momentului de inerţie pentru o sferă rigidă. Efectele de rotaţie determină modificarea formei nucleului în procesele de excitare şi rotaţie. 1,00 puncte

Se acordă **1,00** puncte din oficiu

Neutron beam analysis



a) $L_0 = v\gamma\tau_M^* = \beta c \times \gamma\tau_M^* = \frac{p_M}{m_M} \times c\tau_M^* = \begin{cases} 55{,}7\ m \times \frac{p_\pi}{GeV} \\ 7.48\ m \times \frac{p_K}{GeV} \end{cases}$

with $\beta = \frac{p_M}{E_M}$; $\gamma = \frac{E_M}{m_M}$ and $E_\nu^* = \frac{m_M^2 - m_\mu^2}{2m_M}$

b)

$$P = 1 - \exp\left(-L_D/L_0\right)$$

Using also branching ratio $L_D \simeq 300$ m

The role of muon shield is to absorb the remaining mesons by ionizat and radiation losses.

c)

Energy (E_ν) and angle ($\cos\theta_\nu$) in the laboratory frame are

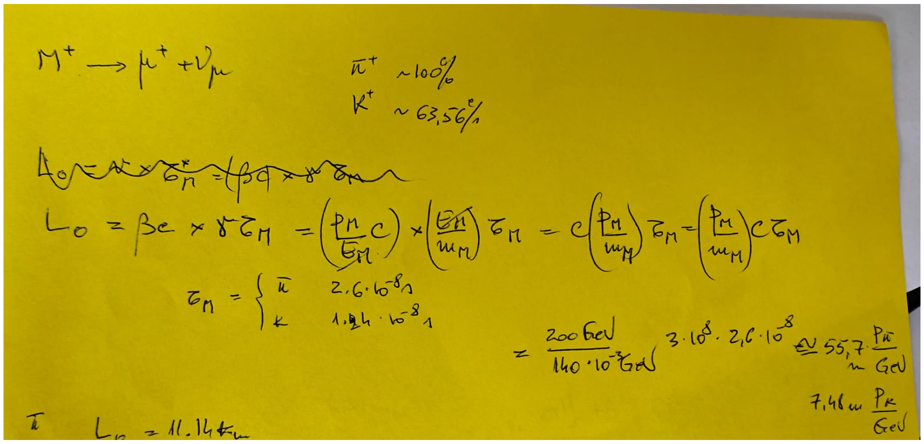
$$E_\nu = \gamma E_\nu^* (1 + \beta\cos\theta_\nu^*)$$

$$\cos\theta_\nu = \frac{\cos\theta_\nu^* + \beta}{1 + \beta\cos\theta_\nu^*}$$

$$E_\nu^{max} = \frac{m_M^2 - m_\mu^2}{2m_M^2}(E_M + p_M) \simeq \left(1 - \frac{m_\mu^2}{m_M^2}\right)E_M = \begin{cases} 0{,}427 \times E_\pi \\ 0{,}954 \times E_K \end{cases}$$

$(E_M \gg m_M)$

$$E_\nu(\theta_\nu) = \frac{m_M^2 - m_\mu^2}{2(E_M - p_M\cos\theta_\nu)} \simeq E_M \frac{m_M^2 - m_\mu^2}{m_M^2 + E_M^2\theta_\nu^2} \simeq E_\nu^{max}\frac{1}{1+\gamma_M^2\theta}$$



$M^+ \longrightarrow \mu^+ + \nu_\mu$ $\pi^+ \sim 100\%$

$K^+ \sim 63{,}56\%$

~~$L_0 = v \times \tau_M^* = (\beta c) \times \gamma\tau_M$~~

$$L_0 = \beta c \times \gamma\tau_M = \left(\frac{p_M}{\cancel{E_M}}c\right) \times \left(\frac{\cancel{E_M}}{m_M}\right)\tau_M = c\left(\frac{p_M}{m_M}\right)\tau_M = \left(\frac{p_M}{m_M}\right)c\tau_M$$

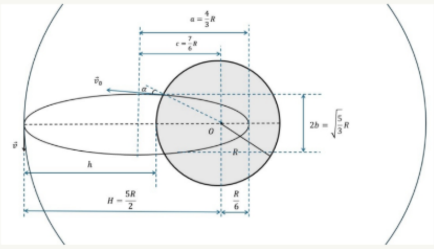
$$\tau_M = \begin{cases} \pi & 2.6 \cdot 10^{-8} s \\ K & 1.24 \cdot 10^{-8} s \end{cases}$$

$$= \frac{200\ GeV}{140 \cdot 10^{-3}\ GeV} 3 \cdot 10^8 \cdot 2{,}6 \cdot 10^{-8} \simeq 55{,}7 \underset{m}{} \cdot \frac{p_\pi}{GeV}$$

$7{,}46\ m\ \frac{p_K}{GeV}$

π $L_D = 11.14$ km

Rocket Launch



2p

a) Conservation of the energy:

$$\frac{1}{2} m v_0^2 - \frac{\gamma m M}{R} = \frac{1}{2} m v^2 - \frac{\gamma m M}{R+h} \quad (1)$$

h- maximum height

v- velocity at the maximum height

At the launching point, the angular momentum is:

$$L_0 = m R v_0 \sin\alpha$$

At maximum height, the velocity vector is perpendicular on radius:

$\Rightarrow L = m(R+h)v$

Proof: In polar coordinates, the velocity is: $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

if $\dot{r} \neq 0$, the distance it is not maximum (or minimum)

$\Rightarrow \cancel{m} R v_0 \sin\alpha = \cancel{m}(R+h)v$

$$v = \frac{R}{R+h} v_0 \sin\alpha$$

Use in (1): $\frac{1}{2}\cancel{m} v_0^2 - \frac{\gamma M \cancel{m}}{R} = \frac{1}{2}\cancel{m}\frac{R^2 v_0^2 \sin^2\alpha}{(R+h)^2} - \frac{\gamma M \cancel{m}}{R+h}$

$$(R+h)^2\left(\frac{v_0^2}{2} - \frac{\gamma M}{R}\right) + \gamma M(R+h) - \frac{1}{2}R^2 v_0^2 \sin^2\alpha = 0 \quad (2)$$

Eq. (2) is a second degree equation from wich may obtain $R+h$

The left side of eq. (2) is a second degree function of $(R+h)$ that has a maximum for $\frac{v_0^2}{2}-\frac{\gamma M}{R}<0$ that corresponds to $\frac{mv_0^2}{2}-\gamma\frac{mM}{R}=E_0<0$ that is an closed orbit (ellipse) around the Earth; it make sense to discuss about maximum (or minimum) distance from Earth.

The two solutions of the equation (2) are possitive because their sum is $-\frac{\gamma M}{\underbrace{\frac{v_0^2}{2}-\frac{\gamma M}{R}}_{<0}}>0$ and the product is $\frac{-\frac{1}{2}R^2v_0^2\sin^2\alpha}{\underbrace{\frac{v_0^2}{2}-\frac{\gamma M}{R}}_{<0}}>0$

$\Rightarrow$ Eq (2) has two solutions corresponding to minimum and maximum distance from the Earth.

b) 1p

Vertical launching $\Rightarrow \alpha=0 \Rightarrow \sin\alpha=0$. Eq. (2) is:

$$(R+h)^2\left(\frac{1}{2}v_0^2-\frac{\gamma M}{R}\right)+\gamma M(R+h)=0$$

$$(R+h)\left[(R+h)\left(\frac{1}{2}v_0^2-\frac{\gamma M}{R}\right)+\gamma M\right]=0$$

$R+h=0$ does not fit

$$R+h=-\frac{\gamma M}{\frac{1}{2}v_0^2-\frac{\gamma M}{R}}>0, \text{ because } \frac{1}{2}v_0^2-\frac{\gamma M}{R}<0$$

$$h=\frac{\gamma M}{\frac{\gamma M}{R}-\frac{v_0^2}{2}}-R=\frac{\cancel{\gamma M}-\cancel{\gamma M}+\frac{v_0^2R}{2}}{\frac{\gamma M}{R}-\frac{v_0^2}{2}}\simeq\frac{\frac{v_0^2R}{2}}{\frac{\gamma M}{R}}=\frac{\frac{v_0^2}{2}}{\underbrace{\frac{\gamma M}{R^2}}_{g}}=$$

$=\frac{v_0^2}{2g}$ familiar result !!!

c) 1p

$$\text{Eq.}(2) \Rightarrow (R+h)^2\left(\frac{1}{2}\frac{5}{4}\frac{\gamma M}{R}-\frac{\gamma M}{R}\right)+\gamma M(R+h)-\frac{1}{2}R^{\cancel{2}}\frac{5}{4}\frac{\gamma M}{\cancel{R}}\frac{1}{4}=0$$

$$R+h=\frac{5}{2}R \Rightarrow h=\frac{5}{2}R-R=\frac{3}{2}R$$

$$R+h=\frac{1}{6}R \Rightarrow h=\frac{1}{6}R-R=-\frac{5}{6}R<0 \text{ does not fit}$$

d) 1p

Since the closest distance from the center of Earth is $R+h=\frac{R}{6}<R$, the rocket will fall on the ground and it can not become a satellite

e) 2p Initial energy of the rocket is:

$$E_0=\frac{m}{2}\frac{5}{4}\frac{\gamma M}{R}-\frac{\gamma m M}{R}=-\frac{3}{8}\frac{\gamma m M}{R}<0 \text{ closed orbit}$$

Evaluate quantity $-\frac{\mu\alpha^2}{2L^2}=-\frac{8}{5}\frac{\gamma m M}{R}<E_0<0$ orbit is permitted and it is an ellipse

Semi major axis: $a=-\frac{\alpha}{2E_0}=\frac{4}{3}R$

Semi minor axis: $b=\frac{L_0}{\sqrt{-2\mu E_0}}=\frac{1}{2}\sqrt{\frac{5}{3}}R$

Focal distance: $c=\sqrt{a^2-b^2}=\sqrt{\frac{16}{9}R^2-\frac{5}{12}R^2}=\frac{7}{6}R$

The biggest distance from the center of the Earth:

$$r_{afe}=a+c=\frac{4}{3}R+\frac{7}{6}R=\frac{5}{2}R \Rightarrow h=r_{afe}-R=\frac{5}{2}R-R=\frac{3}{2}R$$

maximum height

the smallest dist from the center of Earth:

$$r_{per}=a-c=\frac{4}{3}R-\frac{7}{6}R=\frac{R}{6}<R \rightarrow \text{projectile will fall on the Earth}$$

f) 2p At the aphelion, the rocket has

energy $E_0 = -\frac{3}{8}\frac{\gamma m M}{R}$

linear momentum $L_0 = m\sqrt{\frac{5}{16}\gamma M R}$

The manoeuver can be executed by applying a thrust tangential to the orbit at aphelion. Let E be the new energy and L the angular momentum. We assume that the fuel burned by the rocket at transfer is negligible compared with its mass. At apogee, the velocity is tangential and it remains tangential.

The circular orbit is a particular case of ellipticae, thus:

$$\frac{5R}{2} = -\frac{\gamma m M}{2E'} \Rightarrow \qquad E' = -\frac{1}{5}\frac{\gamma m M}{R} \qquad v' = \sqrt{\frac{2}{5}\frac{\gamma M}{R}}$$

$v = \sqrt{\frac{1}{20}\frac{\gamma M}{R}} < v' \Rightarrow$ the ship must accelerate in the tangential direction

$$b' = \frac{L'}{\sqrt{-2mE'}}$$

$$L' = m v' \frac{5}{2}R = m\sqrt{\frac{2}{5}\frac{\gamma M}{R}}\,\frac{5}{2}R = m\sqrt{\frac{5}{2}\gamma M R}$$

$b' = \frac{5}{2}R = a' = R+h$ circular orbit !!!

Total: $2p + 1p + 1p + 1p + 2p + 2p + \underline{1p} = 10p$