PLANCKS 2014

PHYSICS LEAGUE ACROSS NUMEROUS COUNTRIES FOR KICKASS STUDENTS

ANSWER BOOKLET

Utrecht, the Netherlands

23-25 May

A publication of A-Eskwadraat
PLANCKS 2014:

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Date: May 24, 2014

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Table of contents

1 - Graphene

Carlo Beenakker, Leiden University

[1] *2 point* Explain why the conduction electrons in graphene are called “massless particles”.

Answer *The conduction electrons in graphene are called ’massless’ by analogy with photons, which have an energy-independent velocity and zero mass.*

[2] *4 points* Derive, for this case of uniform potential U_0, the relation between the energy E and the wave vector components k_x, k_y. Make a plot of E as a function of $k \equiv \sqrt{k_x^2 + k_y^2}$.

Answer *The plane wave is an eigenfunction of momentum, so we may replace $p_x \mapsto \hbar k_x$ and $p_y \mapsto \hbar k_y$. We therefore seek a nonzero solution of the matrix equation $(H - E)\Psi = 0$, where H is the 2×2 matrix*

$$H = \begin{pmatrix} U_0 & \hbar v k_x - i\hbar v k_y \\ \hbar v k_x + i\hbar v k_y & U_0 \end{pmatrix} \tag{1}$$

Such a solution only exists if the determinant of $H - E$ is zero, which evaluates to

$$Det(H - E) = (U_0 - E)^2 - \hbar^2 v^2 (k_x^2 + k_y^2) = 0 \implies E = U_0 \pm \hbar v k \tag{2}$$

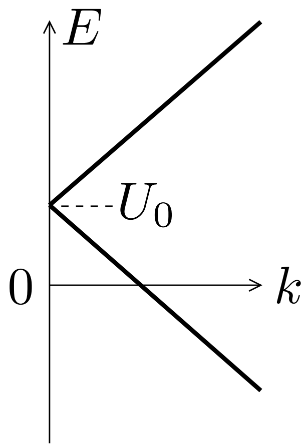

Figure 1: The correct dispersion.
$E = U_0 \pm \hbar v k$

[3] *4 points* Verify this (y-independent) solution of the wave equation:

$$\begin{cases} \Psi_1(x) = C \exp\left(\frac{is}{\hbar v} \int_0^x [E - U(x')]dx'\right) \\ \Psi_2(x) = s\Psi_1(x) \end{cases} \tag{3}$$

with C an arbitrary constant and s equal to $+1$ or -1.

Answer *For motion along the x-axis the wave function has no y-dependence, so the wave equation (1) can be written as*

$$\begin{cases} U(x)\Psi_1(x) - i\hbar v \frac{d}{dx}\Psi_2(x) = E\Psi_1(x), \\ U(x)\Psi_2(x) - i\hbar v \frac{d}{dx}\Psi_1(x) = E\Psi_2(x). \end{cases} \tag{4}$$

The solution (2) can then be verified by substitution.

[4] *3 points* Show that the reflection probability $R = 0$ at any energy no matter how high the potential barrier.

Answer *One way to show that there is no reflection, is to note that the probability density $|\Psi_1(x)|^2 + |\Psi_2(x)|^2 = 2|C|^2$ is independent of x. If R would be nonzero, the probability density should be higher on one side of the barrier than on the other side.*
Alternatively, one can note that far from the barrier, where $U(x)$ tends to a constant $< E$, the

solution $\Psi_2(x) = s\Psi_1(x)$ *describes a wave moving in the positive* x*-direction (for* $s > 0$*) or in the negative* x*-direction (for* $s < 0$*). Because the wave moves in the same direction at both sides of the barrier there is no reflection.*

2 - Newton's Cradle

Jan van Ruitenbeek, Leiden University

[1] *5 points* For $N = 2$ and $N = 3$ describe the set of allowed solutions in N-dimensional velocity space.

Answer *Let v_0 be the velocity of the launched ball before the collision, and v_i the velocity of ball i after the collision. Conservation of momentum gives (taking all masses equal): $v_1 + v_2 + \ldots + v_N = v_0$*
This equation describes a plane in N-dimensional space spanned by the points $(v_0, 0, \ldots, 0)$, $(0, v_0, \ldots, 0), \ldots, (0, 0, \ldots, v_0)$. Conservation of energy gives: $v_1^2 + v_2^2 + \ldots + v_n^2 = v_0^2$
This equation describes a sphere in N-dimensional space, with radius v_0. The solutions are found at the intersections of the plane and the sphere. The solutions are constrained by the requirement that the balls cannot pass through each other, or $v_1 < v_2 < \ldots < v_n$.

$N = 1$ has a single (trivial) solution.

$N = 2$ has two solutions, but the requirement $v_1 < v_2$ leaves only the solution $v_1 = 0,\ v_2 = v_0$

$N = 3$ has infinitely many solutions. They lie on the circle intersecting the sphere with radius v_0 and the plane spanned by the points $(v_0, 0, 0), (0, v_0, 0), (0, 0, v_0)$, but constrained by the requirement $v_1 < v_2 < v_3$.
This leaves all the points on the arc between the point $(0, 0, v_0)$ and the point $(-\frac{1}{3}v_0, \frac{2}{3}v_0, \frac{2}{3}v_0)$ as valid solutions.

[2] *5 points* When we perform the experiment for $N = 3$ we find that only one solution is realised. Which solution is this, and explain why.

Answer *In the experiment we observe that out of the infinitely many solutions only a single solution is realized. For $N = 3$ this is the solution $(0, 0, v_0)$. The explanation must be that the collision proceeds in steps of two-body collisions. The launched ball 1 collides with ball 2, which is a $N = 2$ collision, and which has only a single solution. This ball, in turn, collides with ball 3, which is again a $N = 2$ collision. This leaves the balls 1 and 2 behind at rest and launches ball 3 at velocity v_0.*

3 - 2DEG at the AlGaAs-GaAs interface

Ingmar Swart, Utrecht University

[1] *10 points* Show that the expectation value of the ground state energy using this ansatz is given by:

$$E_g = 2.48 \left(\frac{\hbar^2}{2m^*}\right)^{1/3} F^{2/3} \tag{5}$$

with m^* the effective mass.

Answer *The expectation value for the energy is given by:*

$$E(k) = \frac{\int_0^\infty \psi_0^*(x)|\hat{H}|\psi_0(x)}{\int_0^\infty \psi_0^*(x)\psi_0(x)dx} \tag{6}$$

Using the ansatz wave function given in the exercise, $\psi_0(x) = xe^{-kx/2}$ *and the expression for the Hamiltonian*

$$\hat{H} = \frac{\hbar^2}{2m^*}\frac{\partial^2}{\partial x^2} + Fx \tag{7}$$

one finds:

$$\begin{aligned} E(k) &= \frac{\int_0^\infty xe^{-kx/2}\left[\frac{\hbar^2}{2m^*}(k-k^2x/4)+Fx^2\right]e^{-kx/2}dx}{\int_0^\infty x^2e^{-kx}dx} \\ &= \frac{\int_0^\infty e^{-kx}\left[\frac{\hbar^2 k}{2m^*}x-\frac{\hbar^2k^2}{8m^*}x^2+Fx^3\right]dx}{\int_0^\infty x^2e^{-kx}dx} \\ &= \frac{\frac{\hbar^2k}{2m^*}\frac{1}{k^2}-\frac{\hbar^2k^2}{8m^*}\frac{2}{k^3}+\frac{6F}{k^4}}{\frac{2}{k^3}} \\ &= \frac{\hbar^2k^2}{8m^*}+\frac{3F}{k} \end{aligned} \tag{8}$$

An estimate of the ground state energy can be found using a variational approach: set $\partial E/\partial k = 0$*. This gives:*

$$k = \left(\frac{12m^*F}{\hbar^2}\right)^{1/3} \tag{9}$$

Insert this into the expression for the energy to obtain:

$$E_0 = \left(\frac{\hbar^2}{2m^*}\right)^{1/3} F^{2/3}\frac{9}{2\cdot 6^{1/3}} \cong 2.48\left(\frac{\hbar^2}{2m^*}\right)^{1/3} F^{2/3} \tag{10}$$

4 - Exercises on Particle Physics

André Mischke, Utrecht University

Charged particles in magnetic fields

[1] *2 points* A proton beam with a kinetic energy of 200 GeV passes through a 2 m long dipole magnet with a field strength of 2 T. Calculate the deflection angle Θ of the beam using the relation $2\sin\frac{\Theta}{2} = L/R$ and the momentum $p = 0.3 \cdot qBR$, where L is the length, R the radius, q the charge, and B the magnetic field.
The mass of the proton can be found in the table 1.
Note SI units!

Answer $2\sin\frac{\theta}{2} = \frac{L}{R}$
$p = 0.3qBR$
($m_p = 0.938$GeV/c^2, $E_{kin} = 200$GeV, $B = 20kG = 2$T) in SI system.
Rewriting this gives $\frac{1}{R} = \frac{0.3qB}{p}$.
This gives

$$p^2 = E^2 = (E_{kin} + m)^2 = E_{kin}^2 + 2E_{kin}m \tag{11}$$

Givin:

$$p = (E_{kin}^2 + 2E_{kin}m)^{\frac{1}{2}} \approx 200.936\frac{GeV}{c} \tag{12}$$

Given that $2\sin\frac{\Theta}{2} = L/R$ *we get*

$$2\sin\frac{\Theta}{2} = \frac{0.3 \cdot q \cdot B \cdot L}{2 \cdot p} = \frac{0.3 \cdot 2\ T \cdot 2\ m}{2 \cdot 200.936\frac{GeV}{c}} \tag{13}$$

So $\Theta = 0.38°$

[2] *2 points* In a high energy reaction a proton with a kinetic energy of 10 MeV is bended in a dipole magnet by $10°$ on a length $L = 2$ m. Calculate the necessary field using the relation from part **[1]**.
Answer
$E_{kin} = 10\ MeV$
$\Theta = 10°$
$L = 2\ m$
So

$$2Sin(\frac{\Theta}{2}) = \frac{L}{R} = \frac{0.3 \cdot q \cdot B \cdot L}{p} \tag{14}$$

From what follows that

$$B = \frac{2\ \sin\frac{\Theta}{2} \cdot p}{0.3 \cdot L} \tag{15}$$

Momentum is given by $p = (E_{kin}^2 + 2 \cdot E_{kin} \cdot m)^{\frac{1}{2}} = 137.33\ \frac{MeV}{c}$
Giving:

$$B = \frac{2\ \sin 5° \cdot 137.33\ \frac{MeV}{c}}{0.3 \cdot 2\ m} \approx 36\ mT \tag{16}$$

Particle identification

[3] *2 points* Find an expression for the mass m is a function of the momentum p, the flight length l and the flight time t. Given that the particles move on straight tracks.

Answer *Given that* $\beta = \frac{v}{c}$, $\gamma = (1-\beta^2)^{-\frac{1}{2}}$,$p \cdot c = \beta\ \gamma\ m\ c^2$, $l = 30$ *m and* $c = 3 \cdot 10^8 \frac{m}{s}$

$$m \cdot c^2 = p \cdot c \cdot \frac{1}{\beta\gamma} = p \cdot c \cdot (\frac{c^2 \cdot t^2}{l^2} - 1)^{\frac{1}{2}} \tag{17}$$

[4] *2 points* The distance between the tracking detector and the ToF is l = 30 m. Identify the following particles using the equation obtained in **[3]** and the masses given in table 1 .

Answer

particle 1	$m \cdot c^2$	=	*0.937 GeV*	*proton*
particle 2	$m \cdot c^2$	=	*0.488 GeV*	*kaon*
particle 3	$m \cdot c^2$	=	*0.484 GeV*	*kaon*
particle 4	$m \cdot c^2$	=	*0.141 GeV*	*pion*

Reconstruction of short-lived particles

[5] *2 points* Use the method of *invariant mass* to determine the neutral particles.

Answer

$$m_{inv}^2 = E_i^2 - p_i^2 = E_1^2 + 2E_1E_2 + E_2^2 - p_1^2 - 2p_1p_2 - p_2^2 \tag{18}$$

With $E_i^2 - p_i^2 = m_i^2\ i = 1, 2$

$$m_{inv}^2 = m_1^2 + m_2^2 + 2E_1E_2 - 2p_1 \cdot p_2 = 2|p_1| \cdot |p_2| \cos Angle(p_1, P_2) \tag{19}$$

With $E_i = (m_i^2 + p_i^2)^{\frac{1}{2}}\ i = 1, 2$
For $K \rightarrow \pi^+ + \pi^-\ :\ m_1 = m_2 = m_{pion} = 0.140 \frac{MeV}{c^2}$
$\Rightarrow m_{inv}(\pi^+, \pi^-) = 498 \frac{MeV}{c^2}$

For $\Lambda^0 \rightarrow p + \pi^-$:
$m_{proton} = 0.938 \frac{GeV}{c^2}$
$m_{pion} = 0.140 \frac{GeV}{c^2}$
$\Rightarrow m_{inv}(p, \pi^-) = 1.078 \frac{MeV}{c^2}$

5 - Laser cooling

Dries van Oosten, Utrecht University

[1] *1 point* In the case that the laser is resonant, and in the limit that $s_0 \to \infty$, what is the rate by which the atom scatters photons?

Answer *If $s_0 \gg 1$, that means $P_e = 1/2$, the atoms will have a 50% change of being in the excited state at any time. The scattering rate is thus half the rate of spontaneous emission.* $\Gamma_{\text{sc}} = \frac{1}{2}\frac{\gamma}{2\pi}$.

[2] *1 point* In this case, what is the force exerted on the atom averages over many photon scattering events?

Answer *When an atom absorbs a photon, due to momentum conservation, the momentum of the atom must change by an amount equal to the photon momentum $\hbar\mathbf{k}$, where $\mathbf{k}$ is the wavevector of the laser light and $|\mathbf{k}| = k = \omega/c$. The force on the atom is simply this momentum times the scattering rate. Thus $\mathbf{F} = \Gamma_{\text{sc}}\hbar\mathbf{k} = \frac{1}{2}\frac{\gamma}{2\pi}\hbar\mathbf{k}$. As spontaneously emitted photons are emitted in a random direction, the momentum transfer due to the emission events average out to zero over many events.*

[3] *1 point* What is the force on the atom when $s_0 \ll 1$ and $\delta = 0$?

Answer *When $s_0 \ll 1$, the probability of finding the atom in the excited state is $P_e = s_0/2$. The scattering is then $\Gamma_{\text{sc}} = \frac{1}{2}s_0\frac{\gamma}{2\pi}$ and thus the force is $\mathbf{F} = \Gamma_{\text{sc}}\hbar\mathbf{k} = \frac{1}{2}s_0\frac{\gamma}{2\pi}\hbar\mathbf{k}$.*

[4] *2 points* Now derive the force in the case that $\delta \neq 0$. Allow for the atom to have a velocity $\mathbf{v}$. Make a sketch of the force as a function of $\mathbf{v}$ for the case that $\delta < 0$ and the case that $\delta > 0$.

Answer *When the detuning $\delta \neq 0$ and the atom is moving with a velocity $\mathbf{v}$, the atom effectively sees the laser detuning by $\delta' = \delta + \mathbf{k}\cdot\mathbf{v}$. Then, the off-resonance saturation parameters $s = s_0/(1 + 4(\delta'/\gamma)^2)$ and thus*

$$\mathbf{F} = \frac{1}{2}\frac{\gamma}{2\pi}\frac{s_0\hbar\mathbf{k}}{1 + 4(\delta + \mathbf{k}\cdot\mathbf{v})^2/\gamma^2}$$

[5] *2 points* Now assume there is an identical laser beam counter propagating with the original laser beam. Derive the force. You may neglect the effects of interference between the two beams.
Again, make a sketch of the force as a function of $\mathbf{v}$ for the case that $\delta < 0$ and the case that $\delta > 0$.

Answer *If the atom beam is counterpropagation with the first, it has a wavenumber opposite to the first beam. As we are allowed to neglect the amount of interference, we can simply add the forces due to the two beam. Thus*

$$\mathbf{F} = \frac{1}{2}s_0\frac{\gamma}{2\pi}\hbar\mathbf{k}\left(\frac{1}{1 + 4(\delta + \mathbf{k}\cdot\mathbf{v})^2/\gamma^2} - \frac{1}{1 + 4(\delta - \mathbf{k}\cdot\mathbf{v})^2/\gamma^2}\right)$$

[6] *2 points* Expand the expression you found in **[5]** for small $\mathbf{v}$, such that you have a force that is linear in $\mathbf{v}$. What type of force is this?

Answer *For small* $\mathbf{v}$*, we can Taylor expand* $(\delta + \mathbf{k} \cdot \mathbf{v})^2 \approx \delta^2 + 2\delta\mathbf{k} \cdot \mathbf{v}$*. If we plug this into the line shape we find*

$$\frac{1}{1 + 4(\delta + \mathbf{k} \cdot \mathbf{v})^2/\gamma^2} \approx \frac{1}{1 + 4\delta^2/\gamma^2} - \frac{\gamma^2 \delta \mathbf{k} \cdot \mathbf{v}}{(\gamma^2 + 4\delta^2)^2},$$

which if we plug it into the equation for the force yields

$$\mathbf{F} = -s_0 \frac{\gamma}{2\pi} \hbar \mathbf{k} \frac{\gamma^2 \delta \mathbf{k} \cdot \mathbf{v}}{(\gamma^2 + 4\delta^2)^2}.$$

We call a force dependent on and opposite to the velocity is a friction force.

[7] *1 point* Now, we plug in some numbers for Rubidium-87. The transition frequency for Rubidium is $\omega = 2\pi \cdot 384$ THz and the linewidth $\gamma = 2\pi \cdot 6$ MHz. Calculate the force on the atom in the limit of exercise **[2]** and determine the resulting acceleration. Express the acceleration in terms of the earths gravitional acceleration g.

Answer *The result of* ***[2]*** *was* $F = \frac{1}{2}\gamma\hbar k$*. With* $\omega = 2\pi \cdot 384$ *THz we find a wavelength* $\lambda = 780$ *nm and thus a wavenumber* $k = 2\pi/\lambda \approx 8 \cdot 10^6$ *m*$^{-1}$*. This gives a force of* $F \approx 2.5 \cdot 10^{-21}$ *N. We get the acceleration by dividing the force by the mass, which is 87 atomic mass units. This yields* $a \approx 1.7 \cdot 10^5$ *m/s*^{2}*, which is about 1800 times the earth acceleration.*

6 - No-cloning theorem

Lieven Vandersypen, Delft University of Technology

[1] *1 point* What is the inner product of $|\psi\rangle_1 |\psi\rangle_2$ and $|\phi\rangle_1 |\phi\rangle_2$?

Answer
${}_1\langle\phi|_2\langle\phi||\psi\rangle_1|\psi\rangle_2 = {}_1\langle\phi|\psi\rangle_1 \cdot {}_2\langle\phi|\psi\rangle_2 = \langle\phi|\psi\rangle^2$

[2] *1 point* What is the inner product of $U(|\psi\rangle_1 |s\rangle_2)$ and $U(|\phi\rangle_1 |s\rangle_2)$?

Answer
${}_1\langle\phi|_2\langle s|U^\dagger U|\psi\rangle_1|s\rangle_2 = {}_1\langle\phi|_2\langle s||\psi\rangle_1|s\rangle_2 = {}_1\langle\phi|\psi\rangle_1 \cdot {}_2\langle s|s\rangle_2 = \langle\phi|\psi\rangle$

[3] *2 points* What constraints does this impose on $|\psi\rangle$ and $|\phi\rangle$?
Answer *The constraint is that $\langle\phi|\psi\rangle^2 = \langle\phi|\psi\rangle$. We know that $x^2 = x$ has two solutions, $x = 0$ and $x = 1$. Therefore either $|\psi\rangle = |\phi\rangle$, or $|\psi\rangle$ and $|\phi\rangle$ are orthogonal.*

[4] *2 points* What are the implications for cloning unknown quantum states?

Answer *It is possible to clone one specific state as well as states orthogonal to it, but impossible to clone states that are not orthogonal to each other.*

[5] *3 points* We can also try to clone states using non-unitary processes, including measurements. The first idea that comes to mind is to measure the state of the particle we want to clone, and then to prepare multiple other particles in that same state. Considering spin-1/2 particles, either show that this works, or argue why it doesn't work.

Answer *If a spin-1/2 particle is either in $|\uparrow\rangle$ or in $|\downarrow\rangle$ and we measure its state along z, the measurement returns $\uparrow$ respectively $\downarrow$. We can then proceed to prepare other particles in the same state. However, if the spin-1/2 particle is in some arbitrary state $a|\uparrow\rangle + b|\downarrow\rangle$, measurement still returns either $\uparrow$ or $\downarrow$. Then we have no way of knowing the value of a and b and thus cannot prepare other particles in the state $a|\uparrow\rangle + b|\downarrow\rangle$.*

[6] *1 point* Show that if quantum cloning were possible, it would be possible to communicate faster than light.

Answer *Let's say Alice wishes to transmit a single bit of information. To send a 0, she measures her particle along the z-axis, which will be project it to either $\uparrow$ or $\downarrow$. To send a 1, she measures her particle along the x-axis, projecting it to either $\rightarrow$ or $\leftarrow$. In all cases, Bob's particle is projected to the opposite state of Alice's particle. Bob then makes many copies of his state and measures the particles along the z-axis. If all the measurements give the same outcome, he knows Alice must have measured along z too, and that she sent him a 0. If his measurements give $\uparrow$ or $\downarrow$ with equal probability, he knows Alice must have measured along x and sent him a 1.*
Note: there are variations on this scheme that work too. For instance, Alice can choose to measure along z or not to measure at all, to communicate a 0 or 1 respectively.

7 - Connecting the dots

Henk Blöte, Leiden University

[1] *2 points* Let there be a line connecting point 1 to point $2m$. The remaining points are divided into two groups by this line. On this basis, write down a recursion formula for c_n for general n.

Answer *The remaining groups contain $2m-2$ and $2n-2m$ points. Thus the recursion is*

$$c_n = \sum_{m=1}^{n} c_{m-1} c_{n-m}. \tag{20}$$

[2] *2 points* Use the definition of the so-called generating function

$$P(x) = \sum_{k=0}^{\infty} c_k x^k \tag{21}$$

and the recursion found under part **[1]** to derive an equation that $P(x)$ must satisfy. Solve this equation, which yields $P(x)$ as an explicit function of x.

Answer *Substitution of this recursion in the definition of P(x) gives*

$$P(x) = 1 + x \sum_{k=1}^{\infty} \sum_{m=1}^{k} c_{m-1} x^{m-1} c_{k-m} x^{k-m} \tag{22}$$

The trick is now to interchange the two sums. This yields

$$P(x) = 1 + x \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} c_{m-1} x^{m-1} c_{k-m} x^{k-m} \tag{23}$$

The index of the first and second sums can harmlessly be shifted by 1 and m respectively, which yields

$$P(x) = 1 + x \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} c_m x^m c_k x^k = 1 + xP(x)^2 \tag{24}$$

This is a quadratic equation in P, of which the solutions are

$$P_{\pm} = \frac{1}{2x} \pm \sqrt{\frac{1}{4x^2} - \frac{1}{x}}. \tag{25}$$

Expansion in x, and comparison with the known values of c_n for small n shows that we have to take the minus sign. Thus the desired solution is

$$P(x) = \frac{1}{2x}(1 - \sqrt{1-4x}) \tag{26}$$

[3] *2 points* Using this solution, obtain the first few terms in the Taylor expansion of $xP(x) = \sum_k a_k x^k$. Derive the ratio a_k/a_{k-1} for general k. Write the similar ratio c_n/c_{n-1} as a function of n.

Answer

$$xP(x) = \tfrac{1}{2}(1 - \sqrt{1-4x}) \text{ so that } a_0 = 0;$$
$$\frac{dxP(x)}{dx} = \sqrt{1-4x} \text{ so that } a_1 = 1;$$
$$\frac{d^2xP(x)}{dx^2} = 2(1-4x)^{-3/2} \text{ so that } a_2 = 1;$$
$$\frac{d^3xP(x)}{dx^3} = 12(1-4x)^{-5/2} \text{ so that } a_3 = 2;$$
$$\frac{d^4xP(x)}{dx^4} = 120(1-4x)^{-7/2} \text{ so that } a_4 = 5;$$

The k-th derivative picks up an additional prefactor $2(2k-3)$. Because of the factor $1/k!$ in the Taylor expansion, we have $a_k/a_{k-1} = 2(2k-3)/k$ and $c_n/c_{n-1} = 2(2n-1)/(n+1)$.

[4] *2 points* Give c_n as an explicit function of n.

Answer *Combination of the previous result with the known values of c_n for small n yields the final answer of the problem as $c_n = \frac{(2n)!}{n!(n+1)!}$*

[5] *2 points* The corresponding contribution ΔS to the entropy of a system is $\Delta S = k_B ln(c_n)$ where k_B is Boltzmann's constant. How does ΔS, in leading order, depend on n in the limit of large n?

Answer

$$\Delta S \simeq 2nk_b ln(2) \quad (27)$$

8 - Glaciers and climate change

Michiel Helsen, Utrecht University

[1] *2 points* Find the length of the part of the glacier extending into the ocean, note that L is the glacier length in x-direction (not along the bed).

Answer *The part of the glacier extending into the ocean (ΔL) is the part of the glacier between the part where slope b reaches the sea level or $b = 0$ we call this point L_0, and the maximum length of the glacier, where calving takes place.*
If $b = 0$ it follows that

$$0 = b_0 - sL_0 \Rightarrow L_0 = \frac{b_0}{s} \tag{28}$$

The maximum length L_{max} is reached when the glacier front reaches flotation, i.e. when the buoyancy force from the water depth equals the local weight of the ice.

$$-\rho_{water} b_{L_{max}} = \rho_{ice} H \tag{29}$$

With $b_{L_{max}} = b_0 - sL_{max}$ as the bed at maximum glacies extent, this leads to:

$$L_{\max} = \frac{b_0 + \frac{\rho_{water}}{\rho_{ice}} H}{s} \tag{30}$$

From this follows

$$\Delta L = L_{max} - L_0 = \frac{b_0 + \frac{\rho_{water}}{\rho_{ice}} H}{s} - \frac{b_0}{s} = \frac{\frac{\rho_{water}}{\rho_{i}ce} H}{s} \tag{31}$$

[2] *3 points* Find the solution(s) for the equilibrium length of the glacier, as a function of E.

Answer *The equilibrium solution for the glacier length is case of a calving glacier is obtained when the integrated specific surface mass balance equals the calving rate. The specific surface mass balance can be expressed as:*
$\dot{b}(h) = \beta (b_0 - sx + H - E)$
The calving rate is proportional to the water depth, which we write as: cb_L, hence:
$\int_0^L \beta (b_0 - sx + H - E)\, dx = cb_L$
$-\frac{1}{2} s\beta L^2 + [\beta (b_0 + H - E) + cs] L - cb_0 = 0$
This quadratic equation can be solved using the ABC-formula:
$A = -\frac{1}{2} s\beta$
$B = \beta (b_0 + H - E) + cs$
$C = -cb_0$
$L = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$
Note that this solution is only valid for $L > b_0/s$ (glaciers ending in sea), which is for $E < b_0/2 + H$. For a glacier terminating on land ($E > b_0/2 + H$) the solutions become:
$L = \frac{2(b_0 + H - E)}{s}$ *or* $L = 0$.

[3] *1 point* Ice flows under the influence of gravity.
Considering that the glacier geometry is in equilibrium, at which point do we find the highest

ice velocity?

Answer *When the glacier geometry is constant, the ice flow compensates any imbalance in spatial differences in the mass balance. Therefore ice velocity is largest at the equilibrium line, as the integrated mass balance (upstream of this point) is largest there.*

[4] *3 points* The mean temperature of the atmosphere decreases with height. Assume that E coincides with an isotherm in the atmosphere. Find an expression for the sensitivity of the glacier for a temperature change, i.e. $\frac{dL}{dT}$. The atmospheric temperature gradient is a constant γ.

Answer $\frac{dL}{dT} = \frac{dL}{dE}\frac{dE}{dT} = -\frac{2}{\gamma s}$

[5] *1 point* How does this sensitivity changes when temperatures drop and the glacier front reaches the ocean? Show this qualitatively in a sketch.

Answer *The glacier sensitivity becomes smaller and eventually zero when the glacier tongue reaches the ocean.*

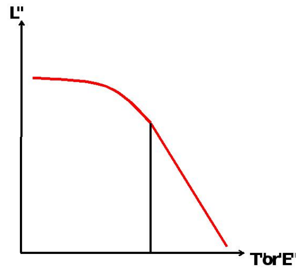

Figure 2: The correct sketch

9 - 24 Hours in a Day – are there?

Gerhard Blab, Utrecht University

[1] *3 points* Argue and sketch how the orbital motion of Earth around the sun leads to a mean solar day that is longer than the rotation of the Earth, and show that this difference should indeed be on the order of 4 minutes. Make sure to clearly indicate directions of rotation!

Answer *The direction of the Earth's rotation and its of motion around the sun are the same. After one rotation* $(1 \rightarrow 2)$ *relative to the fixed stars, Earth has moved also one day's worth of distance along its orbit and the sun has thus not yet reach the local noon. As the part of the orbit that Earth has moved during the day is approximately 1/365 of a circle, the additional rotation that is required must be approximately 1/365 of one day, which turns out to be just under 4 minutes as expected.*

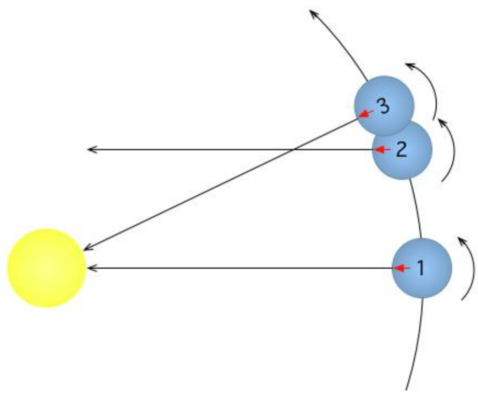

Figure 3: Rotation of the Earth around the Sun. Sizes and distances not to scale
Source: Wikimedia, Gdr

[2] *3 points* Figure shows an "Analemma", a figure obtained by plotting the position of the sun every 24 hours over the course of a year. In it you can find back the seasonal change of altitude, as well as offset between our 24 hour "mean solar day" and the true solar time. Use the figure to estimate the four times during a year at which a day is actually close to 24 hours long, and plot the offset as a function of date (y axis: offset in minutes; x axis: months). This representation is also called the "equation of time".

Answer *The dates should be at times where the tangent of the Analemma is vertical: mid-February (11), mid-May (14), end of July (26) and end-October/early November (4). In order to plot the equation of time, you need to identify 1 degree with* $(24^h = 1440^m)/360 = 4^m$.

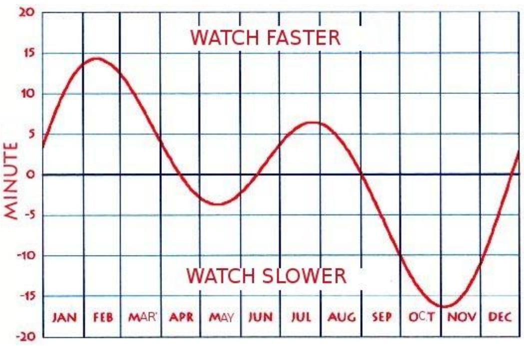

Figure 4: Equation of time, Source: Wikimedia, Willy Leenders, English by Clem Rutter

[3] *4 points* The equation of time is caused by two different effects, both comparable in magnitude: the eccentricity of Earth's orbit and the obliquity ("tilt") of its axis. Explain how those two effects influence the length of a true solar day during a year, and sketch their independent contributions to the equation of time.

Answer *The effect of eccentricity follows directly from Kepler's law (# 2): a line segment between Earth and sun sweeps out equal areas during equal intervals. We are interested in how the angle moved per day changes during the year. The angle itself is proportional to* $1/r \sim (1 + \epsilon \cos\theta)/c$ *(bonus points if you use Kepler's equation), and the contribution to the equation of time is therefore sinosoidal with roots at the apsides in January and July.*
The effect of the obliquity is due to the conservation of angular momentum – the direction of the axis of Earth does not change (on the relevant time scales). This means that the "ecliptic" (the plane in which Earth revolves around the sun, and on which the sun appears to move relative to the fixed stars), will appear to change during the year as the sun moves from its lowest in winter to its highest position in summer. This relative apparent motion was what generated the extra 4 minutes in the first question. The sun moves with constant speed along the ecliptic (remember: we are now considering only obliquity, not eccentricity!), but it does so in different directions: One, which in relative coordinates, is east-west ("right ascention", the direction important for measuring time) and one north-south ("declination", you could also consider this higher or lower in the sky). In conclusion, at the solsices (July and December) the effect of obliquity is maximal – the sun apparently moves east-west, we have to wait longer for Earth's rotation to compensate – and it is minimal at the equinoxes (March and September) when the change in right ascention is smallest. In the graph of the equation of time, we have thus four roots – two with a positive slope at the solistices and two with a negative one at the equinoxes.

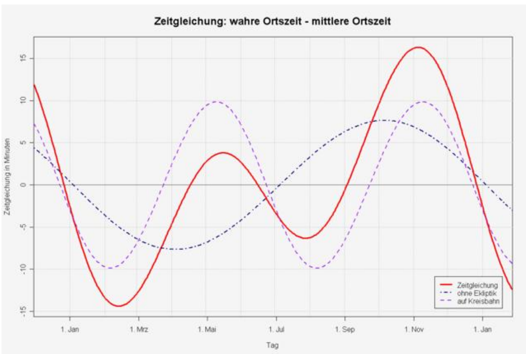

Figure 5: Contributions to the Equation of Time by eccentricity (blue, dash-dot) and obliquity (purple, dash). Note that the figure is flipped relative to figure 4 - there are two ways to define a difference!

10 - Dzyaloshinskii-Moriya interactions and skyrmions

Rembert Duine, Utrecht University

[1] *2 points* Give the configuration of $\mathbf{m}(\mathbf{x})$ with the lowest energy.

Answer *Rewriting $E(m)$ in cartesian coordiants using $m(x) = (m_1(x), m_2(x), m_3(x))$ and $x = (x_1, x_2, x_3)$*

$$E(m) = \frac{J}{2}\int dx((\nabla m_1)^2 + (\nabla m_2)^2 + (\nabla m_3)^2) \tag{32}$$

Obviously, is $m = (c_1, c_2, c_3)$ for fixed $c_1, c_2, c_3 \in \mathbb{R}$, $E(m) = 0$. For any non-constant m, $E(m)$ is positive since the integrand in the equation above is always positive. so m should be constant.

Another method uses Fourier transformation By writing $m = \sum_k m_k e^{ik-x}$ we have $\nabla^2 m = \sum_k -k^2 m_k e^{ik-x}$ Plugg in this into the equation above we get

$$\begin{aligned} &E(m) = \tfrac{J}{2}\int dx \textstyle\sum_{k,k'}(k^2 m_k m_{k'} e^{i(k+k'} \cdot x) \\ &= \tfrac{J}{2}\textstyle\sum_{k,k'} k^2 m_k m_{k'} \int dx e^{i(k+k')\cdot x} \\ &= V\tfrac{J}{2}\textstyle\sum_{k,k'} k m_k m_{k'} \delta_{k,k'} \\ &= V\tfrac{J}{2}\textstyle\sum_k k m_k m_{k'} \end{aligned} \tag{33}$$

with V the volume.
Because m is real $m = m* = \sum_k m_k e^{-ikx}$, we have $m_k = -m_k$, so*

$$E(m) = \frac{VJ}{2}\sum_k k^2 |m_k|^2 \tag{34}$$

For every value of $k \neq 0$, $k^2|m|^2 > 0$, so to minimize the energy we have $m_k = 0$ for $k \neq 0$. So m is constant.

[2] *2 points* Derive the Euler-Lagrange equations for $\mathbf{m}(\mathbf{x})$ by minimizing this energy functional, and show that you obtain

$$J\nabla^2\mathbf{m}(\mathbf{x}) = D\nabla\times\mathbf{m}(\mathbf{x}) \tag{35}$$

Show that a possible solution of this equation is a so-called spin spiral:

$$\mathbf{m}(\mathbf{x}) = \cos(qx)\hat{y} + \sin(qx)\hat{z} \tag{36}$$

and determine q.

Answer *We have*

$$m\cdot(\nabla\times\ m) = m_1\frac{\partial m_3}{\partial x_2} - m_1\frac{\partial m_2}{\partial x_3} + m_2\frac{\partial m_1}{\partial x_3} - m_2\frac{\partial m_3}{\partial x_1} + m_3\frac{\partial m_2}{\partial x_1} - m_3\frac{\partial m_1}{\partial x_2} \tag{37}$$

Now, we vary m slightly: $m \to m + \delta m$
We find

$$\delta E = E(m+\delta m) - E(m) = \int dx(-J\nabla^2 m + D\nabla \times m)\cdot \delta m \tag{38}$$

so $\frac{\delta E}{\delta m} = -J\nabla^2 m + D\nabla \times m = 0$ *with D the Euler-Lagrange equation for* m.
Rearranging gives

$$J\nabla^2 m = D\nabla \times m \tag{39}$$

One can also apply the formula for the Euler-Lagrange equation. By writing

$$\begin{aligned} &E(m) = \int dx \tfrac{J}{2}((\nabla m_1)^2 + (\nabla m_2)^2 + (\nabla m_3)^2) \\ &+\tfrac{D}{2}(m_1(\tfrac{\partial m_3}{\partial x_2} - \tfrac{\partial m_2}{\partial x_3}) + m_2(\tfrac{\partial m_1}{\partial x_3} - \tfrac{\partial m_3}{\partial x_1}) + m_3(\tfrac{\partial m_2}{\partial x_1} - \tfrac{\partial m_1}{\partial x_2}) \end{aligned} \tag{40}$$

we see that the Lagrangian density L is given by:

$$\begin{aligned} &L(m) = \tfrac{J}{2}((\nabla m_1)^2 + (\nabla m_2)^2 + (\nabla m_3)^2) \\ &+\tfrac{D}{2}(m_1(\tfrac{\partial m_3}{\partial x_2} - \tfrac{\partial m_2}{\partial x_3}) + m_2(\tfrac{\partial m_1}{\partial x_3} - \tfrac{\partial m_3}{\partial x_1}) + m_3(\tfrac{\partial m_2}{\partial x_1} - \tfrac{\partial m_1}{\partial x_2}) \end{aligned} \tag{41}$$

Now, we use $\frac{\partial L}{\partial m_i} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\frac{\partial L}{\partial(\frac{\partial m_i}{\partial x_j})}$
$\frac{\partial L}{\partial m_i} = \frac{D}{2}(\nabla \times m)_i)$ *and* $\frac{\partial}{\partial x_j}\frac{\partial L}{\partial(\frac{\partial m_i}{\partial x_j})} = \frac{\partial}{\partial x_j}(2\frac{J}{2}\frac{\partial m_i}{\partial x_j} - \frac{D}{2}m_k\varepsilon_{ijk})$ *with* $k = \{1,2,3\}$ *and* $k \neq i,j$ *and* ε_{ijk} *the Levicevita symbol.*

$$\sum_{j=1}^{3} \frac{\partial}{\partial x_j}\frac{\partial L}{\partial \frac{\partial m_i}{\partial x_j}} = J\nabla^2 m_i - \frac{D}{2}(\nabla \times m)_i \tag{42}$$

so $\frac{D}{2}(\nabla \times m)_i = J\nabla^2 m_i - \frac{D}{2}(\nabla \times m)_i \Rightarrow D\nabla \times m = J\nabla^2 m$
To check $\vec{m}(\vec{x}) = \begin{pmatrix} 0 \\ \cos qx \\ \sin qx \end{pmatrix}$
We have: $\nabla^2 \vec{m} = \begin{pmatrix} 0 \\ -q^2 \cos qx \\ -q^2 \sin qx \end{pmatrix}$ *and*

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} 0 \\ \cos qx \\ \sin qx \end{pmatrix} = \begin{pmatrix} 0 \\ -q\cos qx \\ -q \sin qx \end{pmatrix} \tag{43}$$

So that $Jq^2 = Dq$, $q = D/J$.

[3] *3 points* Show that this winding number is an integer.

Answer *There are two solutions possible.*
First solution:
$\vec{\Omega}_1 \cdot \left(\vec{\Omega}_2 \times \vec{\Omega}_3\right)$ *equals the volume enclosed by* $\vec{\Omega}_1,\ \vec{\Omega}_2$ *and* $\vec{\Omega}_3$
So

$$\vec{\Omega}(x,y) \cdot \left(\vec{\Omega}(x+dx,y) \times \vec{\Omega}(x,y+dy)\right) \simeq dxdy\vec{\Omega}(x,y) \cdot \left(\frac{\partial\vec{\Omega}}{\partial x} \times \frac{\partial\vec{\Omega}}{\partial y}\right) \tag{44}$$

equals the volume enclosed by $\vec{\Omega}(x,y)$*,* $\vec{\Omega}(x+dx,y)$ *and* $\vec{\Omega}(x,y+dy)$ *Since* $\vec{\Omega}$*'s are unit vectors and* $dxdy$ *infinitesimal, this area is* $dxdy$*, which integrated over the unit sphere is* 4π*.*

Second solution:

$$\begin{aligned}
&x(\rho,\phi) = \rho\cos\phi \\
&y(\rho,\phi) = \rho\sin\phi \\
&\tfrac{\partial f}{\partial\rho} = \tfrac{\partial f}{\partial x}\tfrac{\partial x}{\partial\rho} + \tfrac{\partial f}{\partial y}\tfrac{\partial y}{\partial\rho} = \cos\phi\tfrac{\partial f}{\partial x} + \sin\phi\tfrac{\partial f}{\partial y} \\
&\tfrac{\partial f}{\partial\phi} = \tfrac{\partial f}{\partial x}\tfrac{\partial x}{\partial\phi} + \tfrac{\partial f}{\partial y}\tfrac{\partial y}{\partial\phi} = -\rho\sin\phi\tfrac{\partial f}{\partial x} + \rho\cos\phi\tfrac{\partial f}{\partial y}
\end{aligned} \tag{45}$$

Rewriting this gives:

$$\begin{aligned}
\tfrac{\partial f}{\partial y} &= \tfrac{\cos\phi}{\rho}\tfrac{\partial f}{\partial\phi} + \sin\phi\tfrac{\partial f}{\partial\rho} \\
\tfrac{\partial f}{\partial x} &= \cos\phi\tfrac{\partial f}{\partial\rho} - \tfrac{\sin\phi}{\rho}\tfrac{\partial f}{\partial\phi}
\end{aligned} \tag{46}$$

Now write:

$$\vec{\Omega} = \begin{pmatrix} \sin(\theta(\rho,\phi))\cos(\varphi(\rho,\phi)) \\ \sin(\theta(\rho,\phi))\sin(\varphi(\rho,\phi)) \\ \cos(\theta(\rho,\phi)) \end{pmatrix} \tag{47}$$

After some straightforward but long algebra:

$$\vec{\Omega}\left(\frac{\partial\vec{\Omega}}{\partial x} \times \frac{\partial\vec{\Omega}}{\partial y}\right) = \frac{\sin(\theta(\rho,\phi))}{\rho}\left[\frac{\partial\varphi}{\partial\phi}\frac{\partial\theta}{\partial\rho} - \frac{\partial\theta}{\partial\varphi}\frac{\partial\phi}{\partial\rho}\right] \tag{48}$$

and $dx \cdot dy = \rho \cdot d\rho \cdot d\phi$*. We get*

$$W = \frac{1}{4\pi}\int_0^\infty d\rho \int_0^{2\pi} d\phi \sin(\theta(\rho,\phi))\left[\frac{\partial\varphi}{\partial\phi}\frac{\partial\theta}{\partial\rho} - \frac{\partial\theta}{\partial\phi}\frac{\partial\varphi}{\partial\rho}\right] \tag{49}$$

For skyrmion: take $\theta = \theta(\rho)$*. We can now work out the integral:*

$$\int_0^{2\pi} d\phi\frac{\partial\varphi}{\partial\phi} = 2\pi \times \text{integer} \tag{50}$$

$$\int_0^\infty d\rho\sin\theta\frac{\partial\theta}{\partial\rho} = 2 \tag{51}$$

Therefore, W *must be an integer.*

[4] *3 points* Derive the equation of motion for $\theta(\rho)$, starting from the energy that includes a field in the z-direction, i.e., starting from

$$E[\mathbf{m}] = \int d\mathbf{x}\left[-\frac{J}{2}\mathbf{m}(\mathbf{x})\cdot\nabla^2\mathbf{m}(\mathbf{x}) + \frac{D}{2}\mathbf{m}(\mathbf{x})\cdot(\nabla\times\mathbf{m}(\mathbf{x})) - B\mathbf{m}(\mathbf{x})\cdot\hat{z}\right] \tag{52}$$

where B is the magnitude of the field in appropriate units.

Answer
We have:

$$\vec{m}(\phi, z) = m_\theta(\phi, z)\hat{\rho} + m_\phi(\phi, z)\hat{\phi} + m_z(\phi, z)\hat{z} \tag{53}$$

so that

$$m_\theta = 0,\ m_\phi = \sin(\theta(\rho)) \tag{54}$$

and

$$m_z = \cos(\theta(\rho)) \tag{55}$$

We have:

$$\nabla^2_\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) \tag{56}$$

and

$$\nabla\times\vec{m} = -\frac{\partial m_z}{\partial\rho}\hat{\phi} + \hat{z}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho m\psi) \tag{57}$$

Working this out gives:

$$E = 2\pi\int d\rho\,\rho\left(\frac{J}{2}\left[\left(\frac{d\theta}{d\rho}\right)^2 + \frac{\sin^2\theta}{\rho^2}\right] + \frac{D}{2}\left[\frac{d\theta}{d\rho} + \frac{\sin\theta\cos\theta}{\rho}\right] - B\cos\theta\right) \tag{58}$$

Now plug in $\theta + \partial\theta$ for θ and set the variation equal to zero:

$$J\left(\frac{d^2\theta}{d\rho^2} + \frac{1}{\rho}\frac{d\theta}{d\rho} - \frac{\sin\theta\cos\theta}{\rho^2}\right) + D\frac{\sin^2\theta}{\rho} = B\sin\theta \tag{59}$$

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