PLANCKS 2015

Physics League Across Numerous Countries for Kick-ass Students

Exercises

Introduction

Dear contestants,

In front of you, you have the exercises of PLANCKS 2015. You are about to compete for being the best physics student team of the world! We hope you will enjoy the competition. Before you start working on the exercises, a few remarks must be made.

- Teams can consist of 3 or 4 students
- The language used in the competition is English
- The contest consists of 10 exercises, each worth **12 points**. Subdivisions of points are indicated in the exercises.
- **All exercises have to be handed in separately.**
- When a problem is unclear, a participant can ask, via the crew, for a clarification from the jury. The jury will respond to this request. If this response is relevant to all teams, the jury will provide this information to the other teams.
- You are allowed to use a dictionary: English to your native language.
- You are allowed to use a non-scientific calculator.
- The use of hardware (including phones, tablets etc.) is not approved, with exceptions of simple watches and medical equipment.
- No books or other sources, except for this exercise booklet and a dictionary, are to be consulted during the competition.
- The organisation has the right to disqualify teams for misbehaviour or breaking the rules.
- In situations to which no rule applies, the organisation decides.

We wish you all the very best. May the best physics team win!

The organizing committee of PLANCKS 2015

Table of constants

	Name	Value
h	Planck constant	$6.62607 \cdot 10^{-34}$ J s
$\hbar$	Planck's reduced constant	$1.054572 \cdot 10^{-34}$ J s
c	Speed of light	$2.99792458 \cdot 10^{8}$ m/s
k_B	Boltzmann constant	$1.38065 \cdot 10^{-23}$ J/K
N_A	Avogadro number	$6.022141 \cdot 10^{23}$

Contents

1. A Positron in an Electric Field

A positron with mass m, charge e and initial (i.e. at time $t = 0$) velocity $\vec{u}_0 = u_0\vec{e}_y$ is injected into an uniform electric field $\vec{E} = E_0\vec{e}_x$, i.e. at right angles with the electric field lines. We will study the counter-intuitive motion at relativistic speeds and the curved trajectory of the accelerated positron. *Some useful formulas can be found on the next page.*

(1.1) [*1 point*] Show that the relativistic Lorentz factor of the accelerated positron is given by

$$\gamma(t) = \sqrt{1 + \frac{(eE_0t)^2 + p_0^2}{m^2c^2}},$$

where $p_0 = mu_0/\sqrt{1 - u_0^2/c^2}$ is the initial positron momentum and c is the speed of light. Give a detailed derivation with clear motivations for each step.

(1.2) [*1 point*] Derive expressions for the positron velocities $u_x(t)$ and $u_y(t)$ as functions of time in respectively the x-direction and the y-direction.

(1.3) [*1 point*] Make a single plot in which both $u_x(t)/u_0$ and $u_y(t)/u_0$ are shown as function of time. Express time along the horizontal axis in units of $eE_0t/\sqrt{p_0^2 + m^2c^2}$.

(1.4) [*1 point*] Discuss and explain both the initial and the asymptotic behavior of $u_x(t)$ and, in particular, $u_y(t)$, since the latter displays a quite surprising non-Newtonian behavior: the particle is apparently decelerated in a direction in which there is no force.

One of the biggest problems in particle accelerator physics is how to deal with the Coulomb repulsions between particles in the beam. To get a sufficient number of events in colliders, for example, hundreds of millions of charged particles need to be packed into a very compact bunch. For the planned ILC, the successor of the LHC, positrons will be colliding with electrons at a kinetic energy of 0.5 TeV, in bunches hundreds of microns in length and only tens of nanometers across, We will now first investigate how mutual Coulomb repulstion between two positrons travelling side by side in a beam leads to spreading. Subsequently we will see that relativistic effects will help us to deal with the 'Coulomb bomb'.

We first consider the problem in the center-of-mass inertial frame, in which electron 1 is initially ($t = 0$) at position $x_1(t = 0) = x_0$ and electron 2 at position $x_2(t = 0) = -x_0$. Both are initially at rest: $\dot{x}_1(t = 0) = \dot{x}_2(t = 0) = 0$. We assume the velocities the electrons reach *in the center-of-mass frame* are much smaller than the speed of light c, so a non-relativistic treatment (Newtonian mechanics) is allowed.

(1.5) [*2 points*] Show that the time t it takes for the electron initially at position x_0 to reach position x is given by

$$\frac{t}{t_0} = \sqrt{\left(\frac{x}{x_0} - 1\right)\frac{x}{x_0}} + \ln\left(\sqrt{\frac{x}{x_0} - 1} + \sqrt{\frac{x}{x_0}}\right),$$

with $t_0 = \sqrt{\frac{8\pi\epsilon_0 m x_0^3}{e^2}}$. Give a detailed derivation with clear motivations for each step. Hint: use conversion of potential energy into kinetic energy.

(1.6) [*2 points*] Make a plot of $\frac{t}{t_0}$ as a function of x from $x = x_0$ to $x = 5x_0$. Derive an expression for the asymptotic speed v_∞ the electrons reach for $|x| \gg |x_0|$. Add a curve to the plot displaying motion at constant velocity v_∞. For which (numerical) values of x_0 is the non-relativistic treatment justified? Discuss whether a situation might occur in practice where the non-relativistic treatment would not be a good approximation.

(1.7) [*2 points*] Now consider the problem in the lab frame, in which both electrons initially are moving with velocity $v_z \gg v_\infty$ in the z-direction: $x_{1,2}(0) = \pm x_0$, $\dot{x}_{1,2}(0) = 0$, $\dot{z}_{1,2}(0) = v_z$. Derive expressions for the electric field vector and the magnetic field vector due to electron 2 at the position x_1 of electron 1.

(1.8) [*2 points*] At relativistic speeds the problems of Coulomb repulsion are fortunately softened by relativistic effects. Calculate the force exerted by electron 2 on electron 1 as a function of x_1. Using the

fact that $v_z \gg v_\infty$, so that the Lorentz factor may be approximated by $\gamma = 1/\sqrt{1 - v_z^2/c^2}$, show that the repulsion in the lab frame is suppressed by a factor $1/\gamma^2$ with respect to the center-of-mass frame, i.e. compared to non-relativistic, Newtonian intuition. Use basic relativistic arguments, like time dilation and relativistic increase of the mass, to explain this suppression. Discuss why this suppression is essential for the point of collision in the future ILC collider.

Useful formulas

Relativistic equation of motion: $\vec{F} = \dot{\vec{p}} = \gamma m \vec{u}$, with $\vec{F}$ the applied force and $\gamma = (1 - u^2/c^2)^{-1/2} = (1 - \beta^2)^{-1/2}$ the relativistic Lorentz factor. The total energy of the particle is given by $E = \gamma mc^2 = mc^2 + E_{kin}$, with mc^2 the rest energy of the particle and $E_{kin} = (\gamma - 1)mc^2$ the kinetic energy gained through acceleration. Note: this is *not* the familiar Newtonian kinetic energy.

Transformation of electric fields: $\overline{\vec{E}}^{\perp} = \gamma(\vec{E}^{\perp} + \bar{v} \times \vec{B}^{\perp})$, $\overline{E}^{//} = E^{//}$, where the 'bar' reference frame has velocity $\vec{v}$ with respect to the 'non-bar' frame and $\perp$ denotes the electric field component perpendicular to $\vec{v}$ and // parallel to $\vec{v}$. Transformation of magnetic fields: $\overline{\vec{B}}^{\perp} = \gamma(\vec{B}^{\perp} - \vec{v} \times \vec{E}^{\perp}/c^2)$, $\overline{B}^{//} = B^{//}$. Take care of + and − signs!

2. Configurations of DNA Molecules

DNA is a rather stiff polymer. Its mechanical properties can be well approximated by the wormlike chain model, that describes DNA as an elastic rod with fixed contour length and a local bending energy (per length) that is given by $(A/2) \cdot (1/R^2)$. Here A denotes the bending modulus of the rod and R the local radius of curvature.

The shapes of such rods under various boundary conditions have been worked out by Leonhard Euler in 1744, see Figure 1. These shapes, the so-called Euler elasticas, are described by elliptic functions that are difficult to deal with. A useful approximation that typically deviates only about 10% from the exact result is the circle-line approximation. One replaces the exact shape by a set of straight lines and sections of circles that are connected smoothly and minimizes the bending energy with respect to some free parameter. As example consider one half of the lying figure 8 that Euler happened to call Fig. 8 (see Figure 1). One obtains this shape when one bends a beam such that its two ends touch, a relevant problem in gene regulations that involves DNA looping. From a numerical minimization one finds that the apex angle at the tip is given by 81.6°.

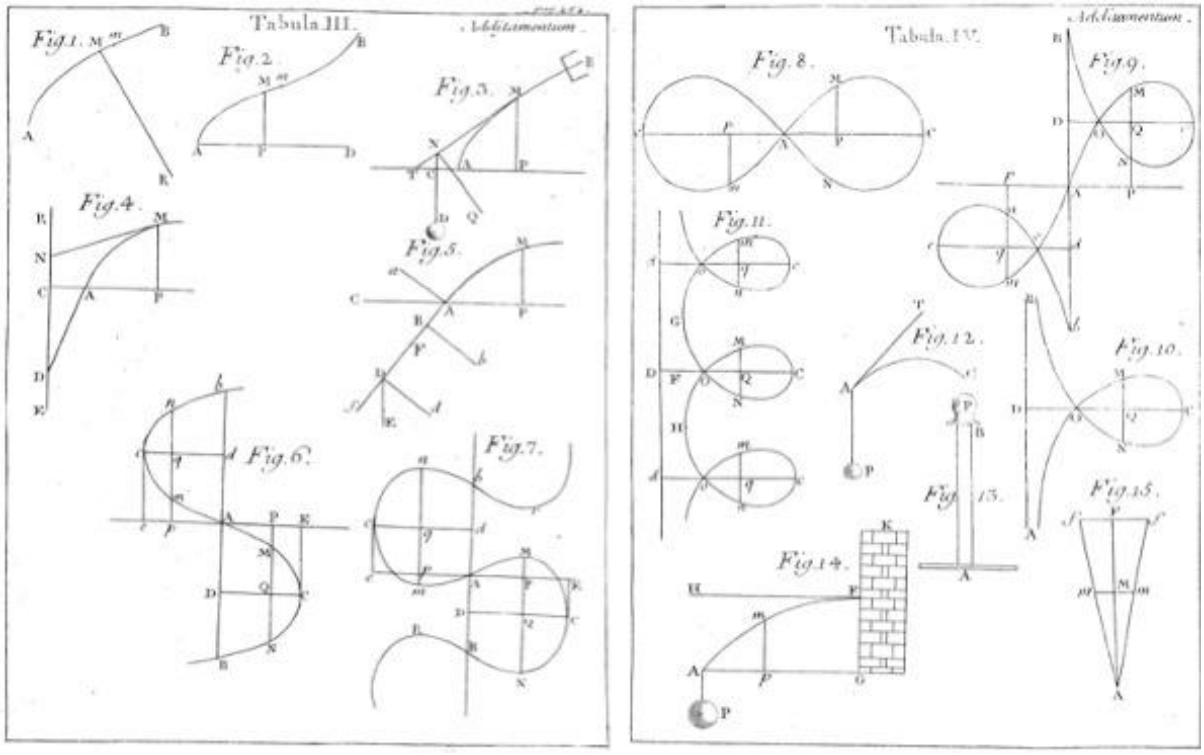


Figure 1: Euler's 1744 drawings of the elasticas

(2.1) [*3 points*] Estimate this angle using a circle-line approximation. You can approximate the tear-shaped loop by two lines that touch at one end and are connected via a circular section at the other end. Assume that the total length of the two lines plus that of the circle is fixed to L. Minimize the bending energy (only the circular part is bent with a constant curvature) with respect to the apex angle. Show that this leads to the transcendental equation $\pi + \alpha = \tan \alpha$ for the tip angle (this is solved for $\alpha = 77.5°$).

We consider next a free DNA molecule, i.e. a molecule without any constraints, but account for thermal fluctuations. One can show that in the thermal avarage - as a consequence of the above mentioned mechanical properties - the tangent-tangent correlations decay exponentially with the distance along the chain: $\langle t(s_0)t(s_0 + x)\rangle = e^{-x/l_p}$, where $l_p = A/k_BT$ is called the *persistence length* (k_BT is the thermal energy). Figure 2 shows the special case $x = l_p$ and states the actual value of l_p for DNA.

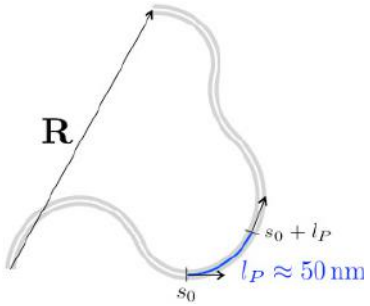


Figure 2: Configuration of a wormlike chain

(2.2) [*3 points*] Calculate $\langle R^2 \rangle$. Hint: use $\vec{R} = \int_0^L \vec{t}(s)\mathrm{d}s$ where L denotes the total contour length of the chain).

We compare the above case now to the case of perfectly flexible polymers that have an enormous number of configuratiuons, all with the same energy. A standard polymer model is the *freely jointed chain*: a chain consisting of N links, each of length b and able to point in any direction independently. A given configuration is then fully characterized by the set of bond vectors $\{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N\}$ (see figure 3).

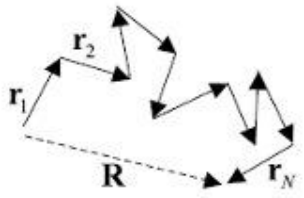


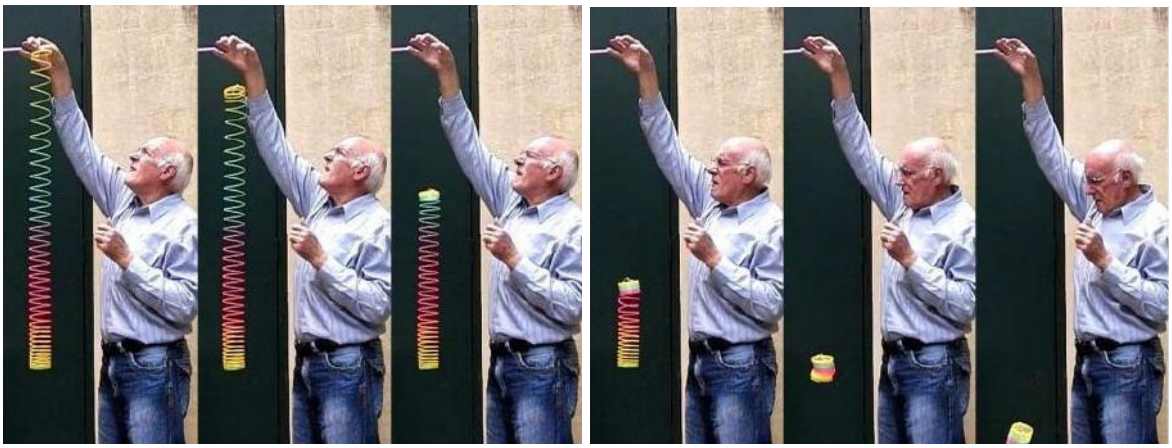
Figure 3: The freely jointed chain

(2.3) [*3 points*] Determine for this polymer model the mean-squared end-to-end distance $\langle R^2 \rangle = \langle \vec{R}^2 \rangle = \left\langle \left(\sum_{i=1}^N \vec{r}_i \right)^2 \right\rangle$. Hint: use the facts that each bond has a fixed length, $\langle \vec{r}_i^2 \rangle = b^2$, and that different bonds are independent from each other, i.e. $\langle \vec{r}_i, \vec{r}_j \rangle = 0$ for $i \neq j$.

(2.4) [*3 points*] Long wormlike chains with $L \gg l_p$ look like flexible chains on large length scales. Show this by comparing the expressions for $\langle R^2 \rangle$ from 2.2 and 2.3.

3. Falling Slinky

Consider a slinky (i.e. a flexible open spring) suspended from its top and at rest. When you release the top end the time evolution of the slinky is fascinating, as shown in the series of pictures below. To describe this phenomenon, we consider an ideal uniform slinky of mass m, and negligible rest length, for which each segment obeys Hooke's law (force $\propto$ extension) $F = kL$. We will consider both its static shape and its dynamic evolution in free fall.



(3.1) [*3 points*] Describe the (vertical) shape of the slinky at rest (left frame). Hint: denote points on the slinky by a dimensionless variable x, ranging from $x = 0$ at the bottom to $x = l$ at the top and describe its shape by specifying the heigth $L(x)$ of each segment above the bottom of the slinky

(3.2) [*1 point*] Explain in words why the slinky behaves the way it does while falling.

(3.3) [*5 points*] How long will it take for the top of the slinky to reach the bottom of the slinky? How does this result compare with the fall time of a small object falling from the same height L?

(3.4) [*3 points*] Derive equations (you don't have to solve them) to describe the distance $\Delta L(t)$ travelled by the top of the slinky at a time t after 'launch', up to moment when it reaches the bottom of the slinky.

4. Measuring Interlayer States in Graphene and Graphite

In a low-energy electron microscope (LEEM), electrons probe the properties of a sample by interacting coherently with the sample's top layers (see Fig. 1). The reflected electrons are collected by a pixel detector to make a spatial image. The energy with which the electrons land, E, is tunable between 0 and 50 eV. These numbers are defined with respect to the so-called vacuum energy E_{vac}, which we take as our zero, i.e. $E_{\mathrm{vac}} = 0$. In vacuum, the electron dispersion relation, relating the electron energy E and the electron wave number k, is given by

$$E(k) = \frac{\hbar^2 k^2}{2m_0},$$

where m_0 is the rest mass of the free electron, and $\hbar = \frac{h}{2\pi}$ with h the Planck constant. By correcting for lens errors (aberrations), the LEEM in Leiden has a lateral resolution (i.e. in the plane) of 1.4 nm at 5 eV.

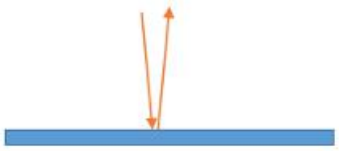


Figure 1

(4.1) [*1 point*] Express the resolution in terms of the wave length lambda at 5 eV. Is there, in principle, room for improvement?

(4.2) [*1 point*] What should happen to the resolution if the electron landing energy approaches 0 eV? What does the latter situation mean, physically?

Interestingly, LEEM does not only have a good lateral resolution. It is also an ideal probe to study the properties of layered materials in the vertical z-direction. The simplest example would be two layers of graphene (graphene is a hexagonal sheet of carbon, one atomic layer thick). In that case, a specific electron state can exist between the two atomic carbon planes. This state is unoccupied, because its energy is above the vacuum energy level. Let us call this state $\Psi_1(z)$ with eigenenergy ε (with respect to E_{vac}). In Fig. 2, you see the result of an experiment in which such a double graphene layer is probed by LEEM. On the x-axis of this so-called $I(E)$-curve, you see the energy of the incoming electrons. On the y-axis, you see the measured intensity I, which is proportional to the electron reflection probability at a certain energy E. One observes a clear dip at energy ε, the eigenenergy of $\Psi_1(z)$. The reason is that at this energy, the incoming electrons couple resonantly to the state $\Psi_1(z)$.

Let us now assume three layers of graphene. To investigate its properties, let us first start with two degenerate interlayer wave functions $\Psi_1(z)$ and also $\Psi_2(z) = \Psi_1(z - c)$, i.e. the same function translated in the z direction over a lattice distance c.

However, these two states can couple quantum mechanically. More exactly, there will be a 'hopping integral' $-t = \langle \Psi_1 | H | \Psi_2 \rangle$, where H is the Hamiltonian and t is a real number.

(4.3) [*2 points*] Express the Hamiltonian H in matrix form assuming the basis (vector) is formed by $(\Psi_1(z), \Psi_2(z))^T$. You may assume that both Ψ_1 and Ψ_2 are normalized.

(4.4) [*2 points*] Solve the time-independent Schrödinger equation, i.e. calculate the eigenvalues of the two resulting interplanar states for triple layer graphene. Express these normalized eigenstates in terms of $\Psi_1(z)$ and $\Psi_2(z)$.

(4.5) [*1 point*] Suppose $t = 1$ eV. Sketch the $I(E)$-curve that one expects to measure for triple layer graphene. Include the portion of the curve you expect to see near and at $E = 0$.

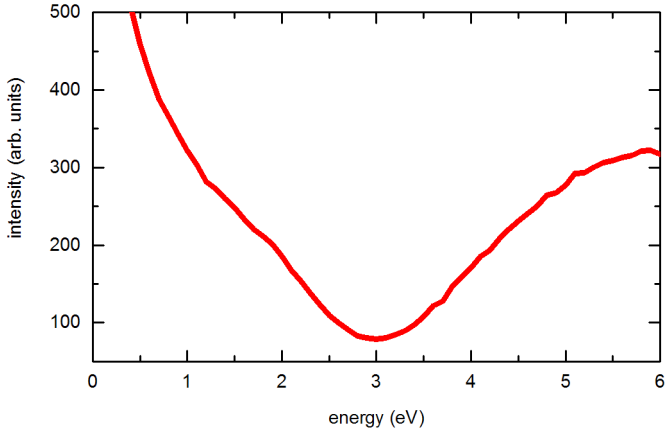


Figure 2

Next, we extend the number of layers of graphene to near infinity. In other words, we take a piece of graphite. To understand what kind of $I(E)$ curve one would measure for graphite, we consider the same interlayer states again. Together they will form a band, with a dispersion relation $\varepsilon(k_z)$, between the electron energy ε and vertical electron wave vector k_z. To calculate $E(k_z)$, we first construct a wave function in k_z space by taking a Fourier transform of the interlayer wave functions in z-space, such that:

$$|k_z\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{ink_z c} |\Psi_n\rangle ,$$

where $|\Psi_n\rangle$ denotes the nth interlayer state.

(4.6) [*2 points*] Calculate the dispersion relation $\varepsilon(k_z)$ by calculating $\langle k_z|H|k_z\rangle$. We assume that only the nearest neighbor interlayer states couple, again with hopping integral -t. Interlayer states that are further away from each other have zero coupling.

(4.7) [*1 point*] Sketch the $I(E)$ curve one expects to measure for graphite. Make clear which energy scales are involved.

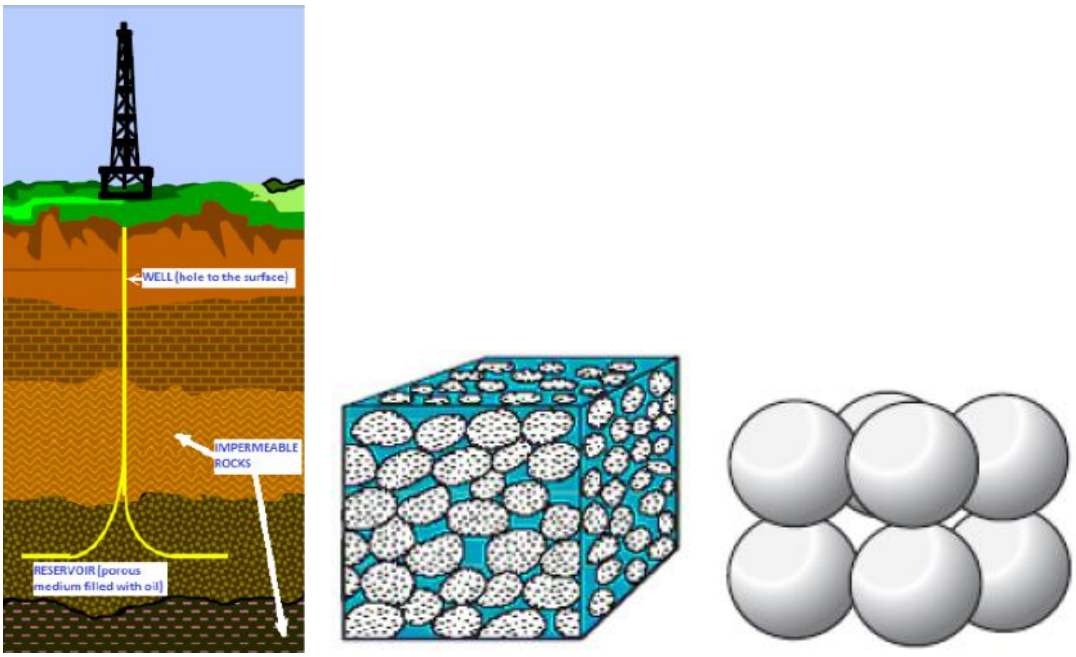
Let us go back to two layers of graphene (see 4.3-4.5). Up to now, we have assumed the electron beam to reach the sample at normal incidence. Suppose we change the angle of incidence from normal towards near-grazing incidence (i.e. the electron comes in almost horizontally).

(4.8) [*2 points*] What will happen to the energetic position of the dip in the $I(E)$ curve of double layer graphene (Fig. 2) as the angle is changed from normal to off-normal? Explain why. (You may give a qualitative answer and use a sketch. However, do explain the basic physics.)

5. Physics of Oil and Gas Production

It is a common misunderstanding that oil is located in the form of an underground lake. Actually, it is found in very small pores with a size comparable to the diameter of the human hair. The void space between the particles of sand is filled with oil like a sponge is filled with water. If one pushes on the sponge, water comes out. A similar thing occurs with oil-saturated rocks, which are located a few kilometers beneath the surface of the Earth. When all the rocks above the reservoir exert a huge pressure on the "petroleum sponge", the oil flows to the surface (see Figure 1a).

In this problem we neglect capillary and gravity effects on the fluid flow.



(a) Scheme of the oil production process

(b) Representation of the porous medium (grains in white and void spacee in blue)

(c) Cubic stacking of identical spherical grains

Figure 1

Basic Concepts

One of the most important characteristics of the reservoir is *porosity*, which is the fraction of the void space in the rock to the total volume:

$$\phi = \frac{V_{void}}{V_{grains} + V_{void}},$$

where V stands for volume.

To understand the meaning of this concept, imagine identical balls (grains of sand) which are put in a pile as shown in Figure 1c.

(5.1) [*1 point*] Find the porosity of the system if the number of balls is infinitely large.

A fluid flow between the grains of sand is controlled by the *viscosity* and the *permeability*. Consider a flow of the viscous fluid through a tube with length L_0 and radius r_0 (see Figure 2). Fluid molecules move along free paths and collide with each other. However, this process is not uniform. Close to the solid boundary the molecules are stuck, while in other regions the velocity varies with a profile similar to the sketch shown in Figure 2, where y starts from the axis of the tube.

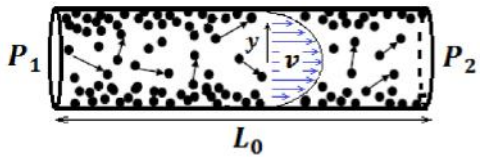


Figure 2: Schematics of the viscous fluid flow in the tube. (Not to scale)

The reason for this effect is the internal friction of the fluid, or viscosity. If two adjacent layers of fluid flow with slightly different speeds, the random sidewise intrusion of some faster molecules into the slower stream will tend to speed up the slower stream, whereas intrusion of slower molecules into the faster stream will tend to slow down the faster stream. This effect could be quantified with the following well-known equation:

$$F_{fr} = -\mu A_{fr} \frac{\mathrm{d}v}{\mathrm{d}y},$$

where F_{fr} is a friction force which occurs between two thin layers of the fluid separated by a small distance $\mathrm{d}y$, which have differences in velocity of $\mathrm{d}v$. A_{fr} is a contact area on which the internal friction force is applied and μ is a fluid property called the *coefficient of viscosity*.

(5.2) [*1 point*] Find the velocity distribution $v(y)$ in terms of μ, L_0, r_0, P_1 and P_2. Assume that a mean free path of molecules is much smaller than the radius of the tube.

Under described conditions fluid will flow through the tube with a flow rate:

$$q = \frac{k_0}{\mu} \pi r_0^2 \frac{P_1 - P_2}{L_0} \qquad \text{(Poiseuille equation)}$$

(5.3) [*1 point*] Find the coefficient k_0 of the Poiseuille equation.

A fluid flow through a porous medium is governed by Darcy's law:

$$q = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{k}{\mu} A \frac{P_{in} - P_{out}}{L},$$

where $\frac{\mathrm{d}V}{\mathrm{d}t}$ is the amount of fluid transferred through the rock in some period of time. A and L are the cross-sectional area and length of the sample shown in Figure 3. $P_{in} - P_{out}$ is the pressure drop and k is the *permeability*, which is the property of the rock. *You can easily recognize some similarity with Fourier's Law for heat transfer. Using analogies with the heat transfer could significantly help you in solving this problem, because the approach is very similar.*

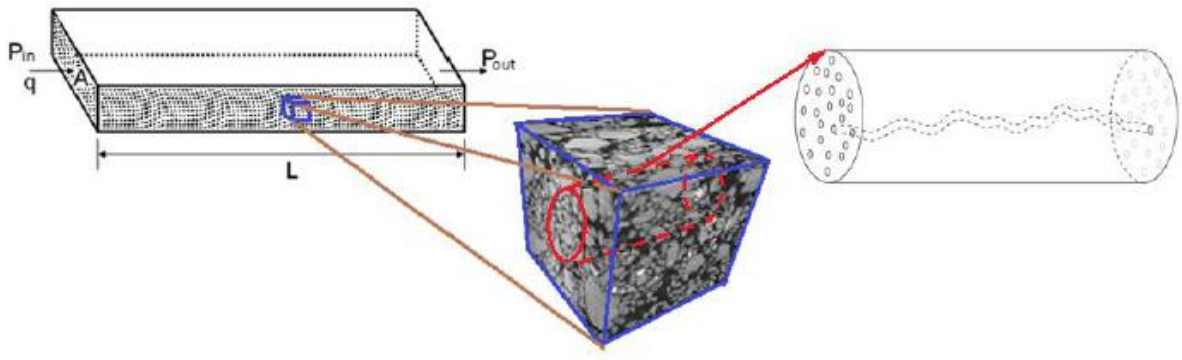


Figure 3: Diagram showing definitions of Darcy's law

The porous medium can be modeled as a system of twisted tubes (Figure 3), with permeability $k = k_0\phi^2$, where k_0 is the permeability of a straight capillary and ϕ^2 accounts for nonlinearity of the tubes in a porous medium with porosity ϕ.

(5.4) [*1 point*] Estimate the permeability of the system described in 5.1, with radii of the balls equal to $1 \cdot 10^{-6} m$.

Usually, rock properties are not uniform throughout the reservoir. However, it is possible to apply an averaging procedure to find an effective permeability k_{eff}. This means that the initial system could be replaced with a "new" model that has the same sizes and fluid flow parameters with the only difference in permeability, which is uniform throughout the "new" homogeneous sample. To examine this issue, we consider a sample consisting of two different rock types as shown in Figure 4. An incompressible fluid flows through that system with a flow rate q and viscosity μ.

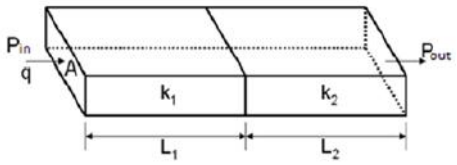


Figure 4: Composite rock sample

(5.5) [*1 point*] Calculate the pressure at the boundary between the two different rocks P_b in terms of q, μ and the parameters shown in Figure 4.

(5.6) [*1 point*] Find the effective permeability of the system k_{eff}.

Vertical Well

Often the reservoir can be modeled as a cylinder (see Figure 5). For this problem all properties were averaged out as in the previous part, so the reservoir is assumed homogeneous with uniform permeability k. Oil can be viewed as an incompressible fluid with viscosity μ. Because the rocks above and below the reservoir are impermeable and the height of the cylinder is much less than its radius ($h \ll R$), one can conclude that the fluid flows only in the radial direction.

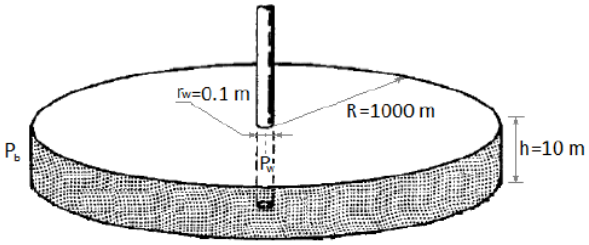


Figure 5: Cylindrical reservoir with a vertical well drilled in the center

(5.7) [*1 point*] Find the velocity of the oil v_w inside the well with a radius $r_w = 0.1$m and a flow rate of $30\text{m}^3/\text{day}$? Estimate the velocity of the fluid in the reservoir near the well.

The calculated fluid velocity in the reservoir is rather small. Therefore the reservoir pressure can be treated as a constant for several months or even years, especially if the reservoir is connected with an underground source of water. Let P_b be the pressure at the outer boundary of the reservoir and P_w be the pressure at the bottom of the well. In this part assume that both P_b and P_w are constants (time-independent values), as well as the radial pressure distribution.

(5.8) [*1 point*] Calculate the pressure drop $P_b - P_w$, which is required to produce oil with a flow rate q.

(5.9) [*1 point*] Make a sketch of the pressure distribution in the reservoir $P(r)$ as a function of the distance from the well.

Modeling Reservoir Depletion

In this part the depletion process will be analyzed for the reservoir shown in Figure 6. The well has a horizontal part, therefore, the fluid flow in the reservoir is linear ($h \ll L$). The fluid can be assumed as incompressible (oil); however, the volume of the reservoir changes during oil production. This is because impermeable rocks around the reservoir squeeze the "petroleum sponge" with their weight. The reservoir rock' *compressibility* c_r can be treated as a constant, which is easily measured and has the following physical definition:

$$c = -\frac{11}{V}\left(\frac{\mathrm{d}V}{\mathrm{d}P}\right)_T$$

where V is the initial volume of the examined sample and $\mathrm{d}V$ is the isothermal volume change, when an additional pressure $\mathrm{d}P$ is applied.

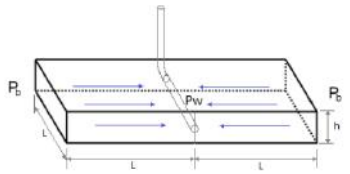


Figure 6: System used for modeling reservoir depletion

This time the pressure at the bottom of the well P_w is constant (hydrostatic column of oil). However, the pressure at the boundary $P_b(t)$ is changing with time, as well as the oil production rate $q(t)$.

(5.10) [*1 point*] Show that $q(t) = \alpha\frac{\mathrm{d}\bar{p}}{\mathrm{d}t}$, where α is a constant in terms of L, h, ϕ and c_r. ϕ is the reservoir porosity, which is independent of time, and $\bar{p}$ is the average pressure at the reservoir, which is a function of time t.

(5.11) [*2 points*] Derive an explicit expression for $q(t)$ in terms of k, μ, c_r, ϕ and the reservoir dimensions, if the initial flow rate is q_0.

6. Scattering

Yukawa potential scattering

Subatomic particles are often studied through scattering processes. Here we will look at one such scattering process, namely due to a Yukawa potential, which has been used as an effective description of massive scalar particle interchange such as the pion. Important calculable observables are the total and differential cross section, which characterise the probability of scattering.

The Yukawa potential is

$$V(r) = V_0 \frac{e^{-\mu r}}{\mu r} \ , \tag{1}$$

where V_0 is an overall normalisation factor which measures the interaction strength and μ is the mass of the force carrier particle. Due to the exponential, the Yukawa potential has a finite range of typical size $R \simeq 1/\mu$. Restricting attention to elastic scattering of a plane wave, one way to derive the cross section for this potential is by using the Born approximation. In the Born approximation it is assumed that the incident wave is not substantially affected by the potential. In terms of the incident plane wave vector $\mathbf{k}$ and outgoing plane wave vector $\mathbf{k}'$, the first-order Born amplitude is given by

$$f^{(1)}(\mathbf{k}, \mathbf{k}') = -\frac{1}{4\pi} \frac{2m}{\hbar^2} \int \mathrm{d}^3 x' \ e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'} \, V(\mathbf{x}') \ , \tag{2}$$

which corresponds physically to a single scattering at the point $\mathbf{x}'$. This simple process yields the largest contribution to the exact scattering amplitude, which means that to good approximation $f(\mathbf{k}, \mathbf{k}') \simeq f^{(1)}(\mathbf{k}, \mathbf{k}')$. As quantum mechanics dictates, the probability of scattering is given by the squared modulus of the amplitude

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\mathbf{k}, \mathbf{k}')|^2 \ . \tag{3}$$

(6.1) [*2 points*] Show that in this case the differential cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \simeq \left(\frac{2mV_0}{\mu\hbar^2}\right)^2 \ \frac{1}{\left(2k^2(1-\cos\theta)+\mu^2\right)^2} \ , \tag{4}$$

where θ denotes the angle between the vectors $\mathbf{k}$ and $\mathbf{k}'$.

(6.2) [*2 points*] A similar-looking *classical* differential cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{mZZ'e^2}{|\mathbf{p}|^2}\right)^2 \ \frac{1}{(1-\cos\theta)^2} \ . \tag{5}$$

Find the corresponding potential.

Inelastic scattering

Inelastic scattering processes are beyond the validity of the Born approximation and we have to resort to other methods, such as the method of partial waves. Far away from the scatterer the quantum mechanical wave function has the asymptotic form

$$\psi(\mathbf{x}) \simeq e^{ikz} + f(\theta)\frac{e^{ikr}}{r} \ . \tag{6}$$

The first term describes a plane wave coming in along the z-axis, while the second term represents an outgoing spherical wave with some unknown complex amplitude $f(\theta)$, which is related to the differential

cross section via $\frac{\mathrm{d}\sigma_{\mathrm{tot}}}{\mathrm{d}\Omega} = |f(\theta)|^2$. The method of partial waves suggests to expand both terms into spherical waves:

$$e^{ikz} \simeq \sum_{\ell=0}^{\infty}(2\ell+1)\frac{1}{2ikr}\Big(e^{ikr} - (-1)^\ell e^{-ikr}\Big)P_\ell(\cos\theta) \quad \text{for large } r \ , \tag{7}$$
$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(k)P_\ell(\cos\theta) \ .$$

The Legendre polynomials $P_\ell(x)$ on the r.h.s. are solutions to the Legendre equation,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P(x)}{\mathrm{d}x}\right] + \ell(\ell+1)P(x) = 0 \ , \tag{8}$$

and are given explicitly in terms of the Rodrigues formula,

$$P_\ell(x) = \frac{1}{2^\ell\,\ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(x^2-1)^\ell \ , \tag{9}$$

where ℓ is a non-negative integer.

(6.3) [*3 points*] Show that the Legendre polynomials are orthogonal, in the sense that

$$\int_{-1}^{1}\mathrm{d}x\ P_\ell(x)\,P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell,\ell'} \ . \tag{10}$$

The quantity of interest is now the complex partial wave amplitude $f_\ell(k)$. It is constrained by the *optical theorem*, which relates the imaginary part of the forward amplitude to the total cross section:

$$\mathrm{Im}f(\theta=0) = \frac{k}{4\pi}\sigma_{\mathrm{tot}} \ . \tag{11}$$

(6.4) [*3 points*] Use the optical theorem to show that $|S_\ell(k)| = 1$, where $S_\ell(k) \equiv 1 + 2ikf_\ell(k)$. What is the physical interpretation of this condition? Hint: write out the formula for the wave function and compare the incoming and outgoing spherical waves.

In the event of absorption by the scatterer this condition must be relaxed. One may then write $S_\ell(k) = \eta_\ell(k)e^{2i\delta_\ell(k)}$, with real $0 \le \eta_\ell(k) \le 1$. In this case the *elastic* cross section is given by

$$\sigma_{\mathrm{el}} = 4\pi\sum_{\ell=0}^{\infty}(2\ell+1)|f_\ell(k)|^2 \ , \tag{12}$$

while the total cross section σ_{tot} is still obtained from the optical theorem.

An example of scattering with absorption is scattering due to a black disc. An incoming wave is incident normally to this disc. The disc has radius R and is capable of total absorption: $\eta_\ell(k) = 0$. Restricting attention to scattering of short wavelengths (large k) limits the angular momentum quantum number of the partial waves to $\ell \le L = kR$.

(6.5) [*2 points*] Compute the elastic cross section as well as the total cross section. Explain the difference.

7. Single Atom Contacts

The context

Electrical resistance for regular, macroscopic wires is the result of scattering of the electrons on defects and lattice vibrations. For a good conductor at room temperature, the average length l an electron travels between scattering events is of the order 50 nm. Remarkably, when the size of a conductor is reduced to dimensions much smaller than this average scattering length l the resistance does not vanish. In this limit quantum mechanics rules.

In this small-size limit it is useful to consider the conductance G rather than the resistance R, where

$$G = \frac{1}{R} = \frac{I}{V},$$

with I the electron current and V the applied voltage. We will assume that all dimensions (lenght L, width W, and thickness D) of the wire are small: $L, W, D << l$. The electrons now behave as waves travelling through a rectangular tube. For a given amount of available energy (the so-called Fermi energy) there is a limited number N of waves accesible. These waves need to fulfill the boundary conditions that the waves have zero amplitude at the surface of the tube. Each of these waves forms a channel for electron transport. The total conductance of the small wire is given as a sum over the contributions of each of the conductance channels,

$$G = \frac{2e^2}{h} \sum_{n=1}^{N} T_n$$

This is the famous Landauer formula. Each channel n that contributes to the conductance is characterized by a number T_n, which gives the probability for an electron entering the channel to exit on the other side. Since T_n is a probability we have

$$0 \leq T_n \leq 1$$

Landauer's formula also shows that each channel contributes a maximum of $2e^2/h$ to the conductance, where e is the electron charge, and h is Planck's constant.

It is possible in experiments to investigate such small conductors. Of special interest are wires that are only one atom thick, as illustrated below. For such atomic wires the number of accesible conductance channels is given by the number of valence orbitals. For simple monovalent atoms such as Cu or Au the wire has just a single conductance channel. In general, the number of relevant channels is larger.

Obviously, by measuring the conductance one only obtains information on the total sum of the transmission probabilities of the available channels.

One way of obtaining more information is by measuring the intrinsic noise in the current. The intrinsic noise, or shot noise, is due to the discrete character of the electron charge. Electrons arriving one-by-one at the other side of the wire produce a current that fluctuates due to the statistical fluctuation in the numbers arriving per given unit of time. This is very similar to the acoustic noise of hail stones falling on a roof. For a small conductor characterized by a set of conductance channels with transmission probabilities $\{T_1, T_2, ..., T_N\}$ the noise power is given by,

$$S = 2eI \frac{\sum_{n=1}^{N} T_n(1-T_n)}{\sum_{n=1}^{N} T_n}$$

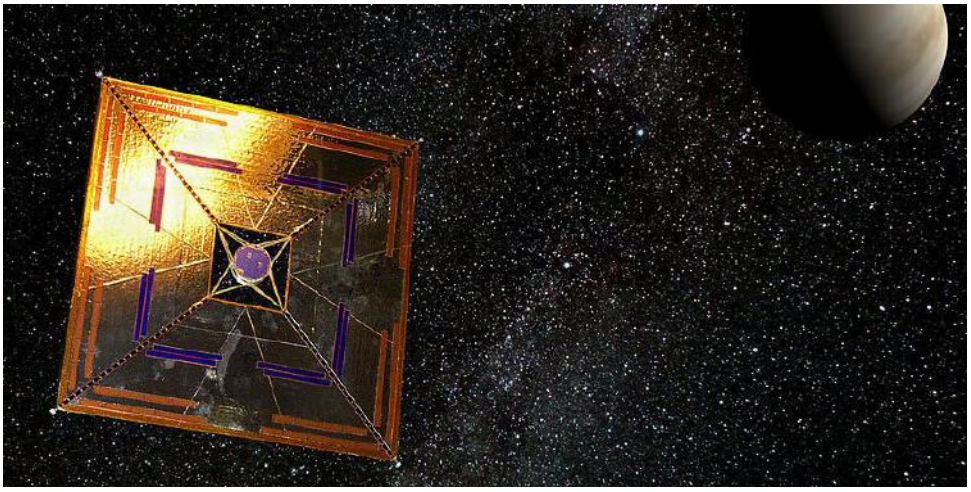
Problems

(7.1) [*6 points*] For a given value of the conductance G, assuming we know there are N channels ($N = 1, 2, 3, ...$), determine the maximum value for the noise power. Also give the values for the transmission values for this solution.

(7.2) [*6 points*] For a given value of the conductance G, assuming we know there are N channels ($N = 1, 2, 3, ...$), determine the absolute minimum for the noise power. Also give the values for the transmission values for this solution.

8. Solar Sail

A wild idea in space technology considers the possibility to propel a space craft by the photon pressure exerted by sunlight. Consider a square sail with dimensions of 800 m $\times$ 800 m and weight $m = 3$ kg that starts (at $t = 0$) from the stationary earth orbit at sufficient distance from the earth not to notice its gravitational pull. Assume that the solar pressure is the only propelling force and that the solar sail is 100% reflective. Call the distance from the solar sail to the sun $r(t)$ and take $r(0) = r_0 = 1.50 \times 10^8$ km. The optical intensity received from the sun (=solar constant) at this distance is $I_0 = 1361$ W/m^2. We will first consider the case where the normal of the sail always points towards the sun. Only in the final question we consider different orientations.



(8.1) [*2 points*] Calculate the initial acceleration a_0 of the solar sail when the normal of the sail points towards the sun and compare this with the acceleration a_c associated with gravity and the centrifugal force.

(8.2) [*1 point*] How long will it *roughly* take the solar sail to cross Mars' orbit at $r = 2.49 \times 10^8$ km $1.66r_0$ if we, as a first rough approximation, assume this acceleration to be constant?

(8.3) [*2 points*] Next, derive the full evolution equation for the distance $r(t)$, using classical mechanics in polar coordinates or something similar. Note: beware of the rotation around the sun.

(8.4) [*2 points*] The evolution equation derived in the previous question is difficult to solve. You can, however, still figure out what the trajectory of the solar sail will look like by comparing this equation with the evolution equation of a moving object that doesn't experience any solar pressure. Based on this comparison, what can you say about (the shape of) the trajectory of the solar sail?

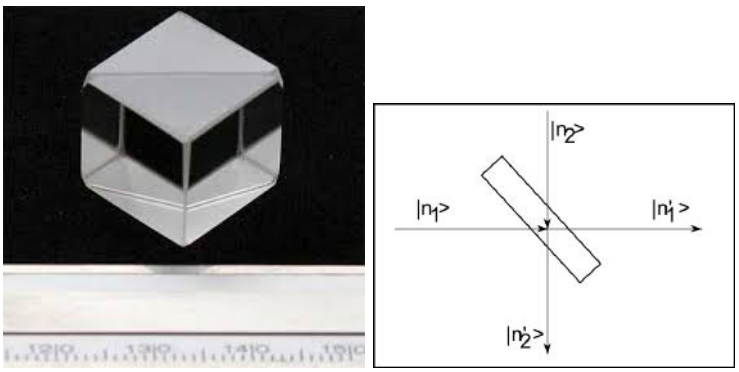
(8.5) [*2 points*] Explain why a solar sail that is 1000$\times$ heavier wouldn't be able to reach mars if the surface normal of the sail remains pointed towards the sun.

(8.6) [*3 points*] Next, also consider different orientations of the solar sail. Which orientation would you approximately choose for efficient escape from the solar system based on optimized increases of the total (i.e. potential + kinetic) energy of the solar sail?

9. The Quantum Mechanical Beamsplitter

The quantum state of light is described by quantizing the harmonic oscilation of the E and B fields, resulting in a set of creation (raising) and annihilation (lowering) operators $a^\dagger$ and a, where each excitation corresponds to a photon.

In this exercise, we will be considering the effect of a beam splitter (e.g. a piece of partially reflecting glass) on such a state of light. We consider the simplest case where there are two input facets, and two output facets. A picture of such a beam splitter is shown below.



We denote the state at the input of the beam splitter $\Psi = |n_1, n_2\rangle$, where the indices refer to the two input facets of the beam splitter. The action of the beam spliter is described by a transformation on the operators

$$a_1^\dagger \to \cos\theta a_1'^\dagger + i\sin\theta a_2'^\dagger, \qquad a_2^\dagger \to i\sin\theta a_1'^\dagger + \cos\theta a_2'^\dagger$$

where the primed operators work on the *output* modes of the beam splitter, as defined in figure 1.

(9.1) [*1 point*] Verify that this operator transformation is unitary. What physical requirement does this correspond to?

(9.2) [*2 points*] Why is it necessary that the number of input ports is equal to the number of output ports? What physical quantity does θ represent?

We now consider the input state $|1, 0\rangle$.

(9.3) [*2 points*] Using the relations given above, compute the state after the beam splitter. What is the probability of finding the photon at either output facet?

We now consider the imput state $|1, 1\rangle$.

(9.4) [*2 points*] Using the relations given above, compute the state after the beam splitter.

(9.5) [*2 points*] What happens when $\theta = \pi/4$? In light of this result, do you agree with Paul Dirac's 1923 statement that photons can only interfere with themselves? Motivate your answer.

We have - without telling you - been considering the rather particular case of *identical* photons. In experiments, however, photons may differ from each other in many ways: wavelength, temporal profile, spatial profile, etcetera.

(9.6) [*3 points*] How would the outcome of the experiment described in 4 change if the photons were *not* identical? Consider the case of two photons differing in their wavelength, called red and blue for simplicity. Hint: in this case, the creation operators should be indexed with the property of the photon.

10. Wind Drift of Icebergs Explained

The Norwegian polar explorer Fridtjof Nansen had observed during an expedition in 1893 that icebergs were not transported in the direction of the wind, but at an angle of 20 to 40 degrees to it. This prompted the Swedish oceanographer Vagn Walfrid Ekman to study this puzzling feature. In 1905 he published his theory that could explain this remarkable finding.

The flow of an incompressible Newtonian fluid is described by the Navier–Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu\nabla^2\vec{v}\,, \tag{1}$$

which describes the evolution in time t of the velocity $\vec{v}$ in a fluid of density ρ and kinematic viscosity $\nu = \mu/\rho$ (μ is the dynamic viscosity); p is the pressure. This equation combined with the constraint of incompressibility

$$\vec{\nabla}\cdot\vec{v} = 0\,, \tag{2}$$

fully describes the fluid motion provided that adequate initial and boundary conditions are given.

Some hints on the applied notations in this equation, using $\vec{v} = (v_x, v_y, v_z)$:

$$\nabla^2\vec{v} = (\nabla^2 v_x, \nabla^2 v_y, \nabla^2 v_z)\,,$$

$$(\vec{v}\cdot\vec{\nabla})\vec{v} = \left(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}, v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z},\right.$$
$$\left. v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right)\,.$$

Ekman found that Earth's axial rotation plays an important role in this problem. Assume a generic rotation vector $\vec{\Omega}$.

(10.1) [*4 points*] Show how to rewrite the Navier–Stokes equations to the following form, representing the motion in the co-rotating reference frame:

$$\frac{\partial \vec{v}}{\partial t} + 2\vec{\Omega}\times\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P + \nu\nabla^2\vec{v}\,,$$

Give also an expression for P, where P is the so-called reduced pressure.

For the remainder you may assume a counterclockwise rotation about the z axis with a positive angular velocity Ω. The dominant force balance in oceanic flow away from the surface and the ocean floor is the so-called geostrophic balance between the pressure gradient and the Coriolis force:

$$2\Omega\hat{\vec{z}}\times\vec{v} = -\frac{1}{\rho}\vec{\nabla}P\,. \tag{3}$$

The vertical unit vector is denoted as $\hat{\vec{z}}$. Under this balance the horizontal velocity components perpendicular to rotation are determined by the pressure gradient.

(10.2) [*3 points*] What can you say about the vertical variation of $\vec{v}$ based on this force balance?

This result is known as the Taylor–Proudman theorem, derived by Joseph Proudman in 1916 and demonstrated experimentally by Geoffrey Ingram Taylor in 1917. The geostrophic balance explains why low-pressure areas in the atmosphere turn counterclockwise in the northern hemisphere, while high-pressure areas turn clockwise. It determines the horizontal velocity components perpendicular to the rotation axis but leaves the magnitude of the vertical component undetermined.

At the interface, however, viscosity must become important as well given that the flow must connect to the airflow above it. The connection is fulfilled by a boundary layer. Assume at $z = 0$ the interface between water and air. This is illustrated in the sketch in figure 1. We will use a reduced equation of

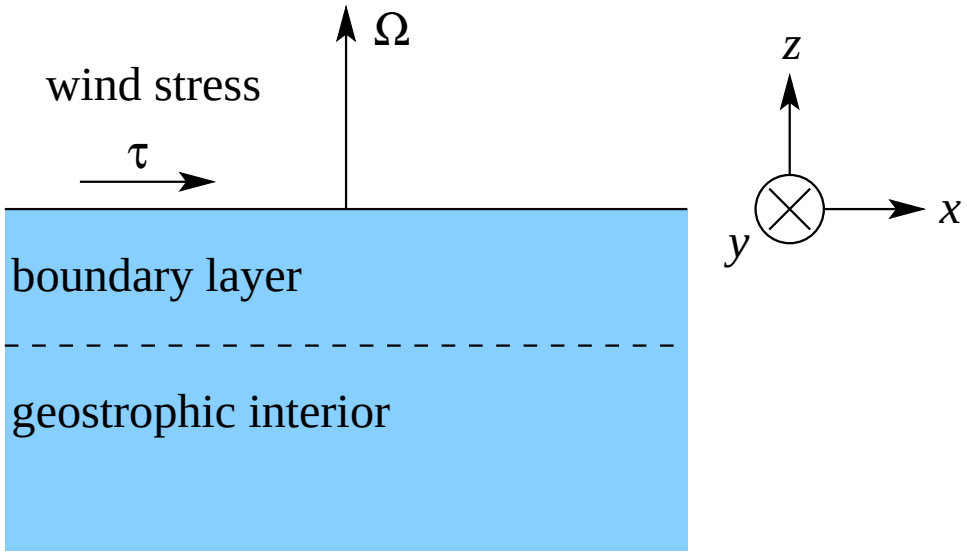


Figure 1: Sketch: wind stress on the ocean surface.

motion which neglects the time derivative and the convective term $(\vec{v}\cdot\vec{\nabla})\vec{v}$ to describe the boundary-layer flow near the wall.

(10.3) [*5 points*] Consider a situation in which horizontal pressure differences are absent. Use the boundary-layer approximation that $\nabla^2 \approx \partial^2/\partial z^2$ to derive the horizontal velocity components v_x and v_y in the boundary layer. The solution should fulfill the following boundary conditions:

$$v_x \to 0\,,\; v_y \to 0 \quad \text{as } z \to -\infty\,,$$

$$\frac{\partial v_x}{\partial z} = \frac{\tau}{\rho\nu}\,,\; \frac{\partial v_y}{\partial z} = 0 \quad \text{at } z = 0\,,$$

where τ is the wind stress, pointing in the x direction. In your solution, what is the predicted drift direction of the icebergs?

The boundary layers derived in this problem are now referred to as Ekman boundary layers, in honor of the person who first described them theoretically.

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