PLANCKS Dublin 2024

Trinity College Dublin
25th May 2024

This exam contains 38 pages (including this cover page) and 10 exercises.
You are required to show your work on each problem on this exam. The following rules apply:

- This paper consists of **10 problems, each worth a total of 10 marks**. For questions with multiple parts, the subdivisions of marks are indicated.
- The exam will **start at 11:30 am**. Participants have a total working time of **four hours** to complete the paper.
- The use of non-medical hardware (including phones, tablets, etc.) and external sources (including textbooks, non-team members, the internet etc.) is not permitted. Scientific, non-programmable and non-graphing calculators, watches and medical equipment are allowed.
- You may use a dictionary: English to your native language.
- Please leave your phones and devices in the envelope at the start to be collected. They will be handed back to you at the end of the exam.
- When a problem is unclear, a participant can ask, via the invigilators, for a clarification. If the response is relevant to all teams, it will be provided to the other teams.
- Organise your work, in a reasonably neat and coherent way. Indicate at the top of each page your UIC, the question number, and the page number for that question.
- Place all problems in the envelope in order at the end of the exam. If your answer to a question has multiple pages, please staple the pages together for each question.
- If it is brought to the invigilators' attention that a team has been cheating, breaking the rules or misbehaving, they will be disqualified. Disqualification can happen post submission.
- In situations to which no rule applies, the organisation will rule on the matter following consultation with the UK & Ireland's PLANCKS Advisory Board, who serve as the jury for PLANCKS Dublin 2024.

PLANCKS Dublin 2024 Exam

Question 1: Moment of Inertia

Question 2: Quaternions

Question 3: Möbius Strip.... Time Travel?

Question 4: Icy Roof

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Question 6: Nuclear Medicine Scan

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Question 9: Molecular Zipper

Question 10: 4D Sun

In case you have forgotten for this **PLANCKS** exam: Planck's constant, $h = 1$. However, if you need it in actual experimentalists' units $h = 6.63 \times 10^{-34}$ Js.

1 Moment of Inertia

This question concerns the moment of inertia, I_{G}, of a uniform lamina and mass m, about an axis perpendicular to the plane of the lamina, passing through the mass centre, G.

Figure 1 shows ABC is an isosceles triangle with two sides length r and third side $2a$.

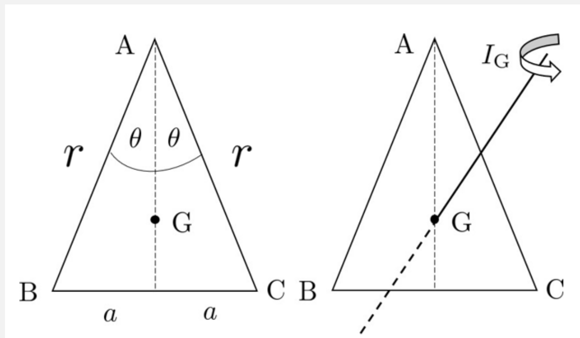


Figure 1: Isosceles triangle with semivertical angle θ and moment of inertia I_G.

(a) [6 marks] Show that the formula for the moment of inertia about the centre of mass is

$$I_{\mathrm{G}} = \frac{1}{18} m r^2 \left(2 - \cos 2\theta\right).$$

Consider now a regular $n-$gon lamina comprised of n isosceles triangles. The diameter of the $n-$gon will be $2r$ and its mass centre is denoted as G'. You may take the total mass of the $n-$gon lamina to be M.

For example, a $6-$gon lamina (i.e a hexagon) would be formed of six isosceles triangles, with their apexes touching. Each of these triangles has two sides of length r.

(b) [4 marks] Find an expression for the moment of inertia of a regular $n-$gon lamina about its centre of mass G' and show that in the limit $n \to \infty$ the result for a circle is recovered i.e.

$$\frac{1}{2} M r^2.$$

Moment of Inertia - Solution

We will take G to be at the origin, with the x-axis parallel to BC and the y-axis along the median. We will also set the height of the triangle to be $3h$, therefore we have $r\cos\theta = 3h$. The moment of inertia is given by

$$I_{\mathrm{G}} = \int_A d^2 \, \mathrm{d}A$$

where d is the distance in the plane from G and the integral is taken over the area A. Note, by symmetry, we need to consider only one half of the triangle with the parameter h.

The equation of the line AC is easily found with our chosen axes to be

$$y = \frac{h}{a}\left(2a - 3x\right)$$

and hence

$$I_{\mathrm{G}} = 2\sigma \int_0^a \mathrm{d}x \int_{-h}^{\frac{h}{a}(2a-3x)} \left(x^2 + y^2\right) \mathrm{d}y.$$

Evaluating gives us,

$$I_{\mathrm{G}} = 2\sigma h a \left[\frac{a^2}{4} + \frac{3h^2}{4}\right]$$

writing this in terms of the mass of the triangle we get

$$I_{\mathrm{G}} = \frac{1}{6} m \left(a^2 + 3h^2\right).$$

Using $r\cos\theta = 3h$ and our trigonometric identities we can rewrite this in the form required by the question

$$I_{\mathrm{G}} = \frac{1}{18} m r^2 \left(2 - \cos 2\theta\right).$$

A regular $n-$gon of diameter $2a$ comprises n isosceles triangles of the type shown in figure 2, G$'$ is the mass centre of the $n-$gon lamina. Note we have set the total mass of the $n-$gon to be M.

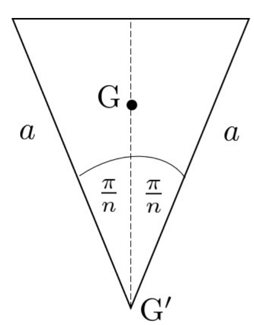


Figure 2: One isosceles triangle of the regular $n-$gon.

For this triangle

$$I_{\mathrm{G}} = \frac{1}{18}\left(\frac{M}{n}\right)a^2\left(2-\cos\left(\frac{2\pi}{n}\right)\right).$$

By the parallel axes theorem, for this triangle,

$$I_{\mathrm{G}'} = I_{\mathrm{G}} + \left(\frac{M}{n}\right)d^2, \text{where } d = \mathrm{GG}'.$$

Since the mass centre, G, of the lamina is the centroid of the isosceles triangle, it follows that

$$d = \frac{2}{3}a\cos\left(\frac{\pi}{n}\right).$$

The $n-$gon comprises n such triangles so that the moment of inertia is given by

$$I_{\mathrm{G}'}^{(n)} = n\left(I_{\mathrm{G}} + \left(\frac{M}{n}\right)d^2\right)$$

and hence

$$I_{\mathrm{G}'}^{(n)} = Ma^2\left[\frac{1}{18}\left(2-\cos\left(\frac{2\pi}{n}\right)\right) + \frac{4}{9}\cos^2\left(\frac{\pi}{n}\right)\right].$$

We can simplify this significantly to

$$I_{\mathrm{G}'}^{(n)} = \frac{Ma^2}{6}\left(1+2\cos^2\frac{\pi}{n}\right)$$

and in the limit $n\to\infty$ the cosine squared term will tend to 1 which yields our desired result for the moment of inertia for a disc,

$$\frac{1}{2}Mr^2.$$

2 Quaternions

Everybody[a] knows how complex numbers work. Each complex number is made of a doublet of real numbers:

$$\begin{aligned} z_1 &= a_1 + ib_1 \\ z_2 &= a_2 + ib_2 \end{aligned}$$

where $i^2 = -1$. In order to be useful, one has to be able to add and multiply complex numbers:

$$\begin{aligned} z_1 + z_2 &= (a_1 + a_2) + i(b_1 + b_2) \\ z_1 z_2 &= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2). \end{aligned}$$

Fundamentally, in order to be considered addition and multiplication, these definitions should satisfy standard associativity and distribution rules

$$\begin{aligned} (z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) \\ (z_1 z_2) z_3 &= z_1 (z_2 z_3) \\ z_1 (z_2 + z_3) &= z_1 z_2 + z_1 z_3, \end{aligned}$$

which can easily be shown to be true for complex numbers. In addition, one would like to be able to associate a complex number with a real number known as its magnitude

$$|z_1|^2 = a_1^2 + b_1^2,$$

which amongst other things should satisfy the rule

$$|z_1 z_2| = |z_1||z_2|.$$

In 1843, Hamilton wanted to extend ‘couples’ (complex numbers) to ‘triples’ – numbers of the form

$$z = a + ib + jc$$

where $i^2 = -1$ and $j^2 = -1$ (but $i \neq j$). The idea is that if you can make up one ‘imaginary’ square root of -1, why not make up another in a ‘perpendicular’ direction. Unfortunately reality turned out to be less enthusiastic about this idea. Apparently, Hamilton’s children would ask him at breakfast each morning *“Papa, can you multiply triples yet?”* to which he would reply sadly *“No, I can only add and subtract them”*. In October of that year while out for a walk, the answer finally came to him. He graffitied the answer into the Brougham Bridge. The graffiti has faded over the years, but there is now a plaque commemorating the event if you wish to visit it while in Dublin.

[a] At least, everybody who makes it to a PLANCKS final

(a) [1 mark] Prove that $ij = -ji$, i.e. the different square roots of -1 must anti-commute.

(b) [2 marks] Prove that if one writes $k = ij$, then $k^2 = -1$, i.e. k is another square root of -1. Further prove that k can not be written as a ‘triple’ i.e. $k \neq \alpha + \beta i + \gamma j$ for real α, β, γ.

You have now shown that 'triples' do not exist if one wants a consistent algebra including multiplication – they must be extended into quaternions.

Hamilton had a notation for quaternions

$$q = (s, \vec{v}) = s + v_x i + v_y j + v_z k,$$

where he called the real part of the quaternion s the 'scalar' part, and the imaginary parts of the quaternion $\vec{v}$ the vector part. In this notation, consider $q_1 = (0, \vec{v}_1)$ and $q_2 = (0, \vec{v}_2)$.

It is worthy of historic note that the modern concept of scalar and vector product had not yet been invented when Hamiltonian first developed his theory of quaternions.

(c) [2 marks] From the rules you have previously derived about multiplying quaternions, prove that

$$q_1 q_2 = (-\vec{v}_1 \cdot \vec{v}_2, \vec{v}_1 \times \vec{v}_2).$$

(d) [5 marks] Calculate e^q if

$$q = \frac{\pi}{5} i + \frac{4\pi}{15} k.$$

Hint: First, work out why this is not trivial. Second, you may find it easier to first work out the general formula for e^q where q is a quaternion, and then substitute in the specific value given.

Quaternions - Solution

(a) (1 mark) Consider a triple

$$z = ia + jb.$$

We could include a real component in this number, but we don't need to for the proof. We are going to assume associativity and distribution laws (as the questions says we must), so the square

$$z^2 = (ia + jb)(ia + jb) = i^2a^2 + j^2b^2 + ab(ij + ji). \tag{1}$$

Notice we haven't assumed commutivity – indeed the question is asking us to prove that multiplication is not commutative. Using the rule that $i^2 = j^2 = -1$, we have

$$z^2 = -(a^2 + b^2) + ab(ij + ji)$$

Now, by the rules of the modulus,

$$|z|^2 = a^2 + b^2$$

We would like $|z|^2 = |z^2|$. We can see from Eq. (1) that this will only be true if $ab(ij+ji) = 0$, i.e. $ij = -ji$ as this must hold for all real numbers a and b.[1]

(b) (2 marks) Let $k = ij$. Then

$$k^2 = (ij)(ij) = i(ji)j = -i(ij)j = -(ii)(jj) = -1 \cdot -1 \cdot -1 = -1$$

where we have used the distributive law of multiplication, along with first $ij = -ji$ and then $i^2 = j^2 = -1$.

We now suppose $k = \alpha + \beta i + \gamma j$. Then

$$k^2 = \alpha^2 - \beta^2 - \gamma^2 + 2\alpha(\beta i + \gamma j).$$

The only way k^2 can be real and negative is if $\alpha = 0$ - so k has no real component.

We know that $ijk = -1$, because $ij = k$ and $k^2 = -1$. Taking $k = \beta i + \gamma j$, we get

$$jk = -\beta ij - \gamma$$

where we have used $ij = -ji$ and $j^2 = -1$. Hence

$$ijk = \beta j - \gamma i.$$

This is clearly not equal to -1 for any value of β and γ, hence k can not be written as a triple.

(c) (2 marks) We begin by summarising how to multiply quaterions:

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

We already derived the first two of these relations; the final ones follow trivially from these first ones (e.g. $jk = jij = -j^2 i = i$, etc.)

To derive the multiplication rule shown, there is no trick, you just multiply it out. Simplifying

[1] If one wants to be more mathematically rigorous, write $ij + ji = \alpha + \beta i + \gamma j$ and show that α, β, γ must all be zero. For the purpose of physics though, I think this is 'obvious' enough.

notation by writing $v_1 \equiv v$ and $v_2 \equiv w$:

$$q_1 q_2 = (v_x i + v_y j + v_z k)(w_x i + w_y j + w_z k) \tag{2}$$
$$= v_x w_x i^2 + v_x w_y ij + v_x w_z ik + v_y w_x ji + v_y w_y j^2 + v_y w_z jk + v_z w_x ki + v_z w_y kj + v_z w_z k^2 \tag{3}$$
$$= -(v_x w_x + v_y w_y + v_z w_z) + (v_y w_z - v_z w_y)i + (v_z w_x - v_x w_z)j + (v_x w_y - v_y w_x)k \tag{4}$$

where in the final step we used the basic rules above. This answer is equivalent to the answer in vector notation given in the question. It is worthy of historic note that the vector product (which is now familiar to us) was actually introduced after Hamilton discovered quaternions. One of the earliest applications of quaternions in physics was in writing Maxwell's equations - which are now written using vectors.[2]

(d) (5 marks) Let $q = s + ia + jb + kc$. Then

$$e^q = e^{s+ia+jb+kc} = e^s e^{ia+jb+kc} \neq e^2 e^{ia} e^{jb} e^{kc}.$$

The first equality follows because the real part s commutes with the rest of quaternion; however we can't split the exponential further because i, j, k do not commute with each other. Those of you familiar with the Baker-Campbell-Hausdorff (BCH) formula may be tempted to use this:

$$e^{X+Y} = e^X e^Y e^{-[X,Y]/2}$$

however this is only true if the commutator $[X, Y]$ is a 'c-number', i.e. commutes with X and Y. In the case of quaternions, this is not the case – $[i, j] = 2k$, but k also does not commute with i or j. There are extensions of the formula involving repeated commutators which will eventually get you to an answer, however this is not usually easy to use (assuming you remember it at all!)

The best way to proceed is to calculate by hand. We have already seen that the real part of a quaternion factors out, so let us consider $s = 0$ and write $q = ia + jb + kc$. Then by definition,

$$e^q = 1 + q + \frac{q^2}{2!} + \frac{q^3}{3!} + \dots.$$

Now $q^2 = -a^2 - b^2 - c^2$ which is real[3] – let us therefore write $q^2 = -|q|^2$. Hence:

$$q^2 = -|q|^2 \tag{5}$$
$$q^3 = -|q|^2 q = -|q|^3 \frac{q}{|q|} \tag{6}$$
$$q^4 = |q|^4 \tag{7}$$
$$q^5 = |q|^4 q = |q|^4 \frac{q}{|q|} \tag{8}$$

and so on. Therefore

$$e^q = (1 - \frac{|q|^2}{2!} + \frac{|q|^4}{4!} - \dots) + \frac{q}{|q|}(|q| - \frac{|q|^3}{3!} + \frac{|q|^5}{5!} - \dots)$$

[2] Or tensors for the purists.

[3] derived by explicit calculation

As $|q|$ is real, we can resum the series:

$$e^q = \cos(|q|) + \frac{q}{|q|}\sin(|q|).$$

One could now write a nice final answer for the exponential of a quaternion including a real part, if one were of such a disposition.

For the question given there is no real part, and $|q| = \pi\sqrt{(1/5)^2 + (4/15)^2} = 5\pi/15 = \pi/3$. Hence

$$e^q = \cos(\pi/3) + \frac{\pi}{5}(3i + 4j)\sin(\pi/3) = \frac{1}{2} + \frac{3\sqrt{3}\pi}{10}i + \frac{4\sqrt{3}\pi}{10}k.$$

3 Möbius Strip.... Time Travel?

In Avengers Endgame, Tony Stark asks his AI assistant 'Friday' to find the eigenvalues of an inverted Möbius strip so that they can build a device to time travel into the past and... *spoilers.*

What is a Möbius strip? Imagine a rectangle of length L and width W, you connect the two ends and the result is a band of circumference L. However, if you **twist** the rectangle 180° before you connect the ends, you get a Möbius strip.

In this problem, you will solve the Schrödinger equation for a quantum particle of mass m confined on a Möbius strip.

For this problem, consider the easiest geometry that a space with Möbius strip topology can have- a flat space with Möbius strip boundary conditions. The problem will be further simplified by ignoring spin.

(a) [6 marks] Find the energies and normalised wavefunctions of a quantum particle of mass m moving on a Möbius strip of length L and width W with potential $V(x, y) = 0$ everywhere on the strip.

Now let us imagine that the confined quantum particle is an electron with mass, $m = 9.11 \times 10^{-31}$ kg.

The electron is described at time $t = 0$ by a wave packet with wavefunction confined to a circle of radius a at the point $x = \frac{W}{2}$, $y = 0$ (depending on your orientation of the strip).

Take the length of the Möbius strip, $L = 20$ nm and the width, $W = 3$ nm.

(b) [4 marks] After what time will the wavefunction return to its initial position? Give your answer in seconds.

State any assumptions you have made to get to your answers.

Möbius Strip.... Time Travel? - Solution

We will take the width of the Möbius strip to be the x-direction and the length with the twist to be the y-direction. Our boundary conditions are:

$$\psi\left(x=0,y\right)=\psi\left(x=W,y\right)$$

and

$$\psi\left(x,y+L\right)=\psi\left(W-x,y\right).$$

The plane wave solution compatible with our boundary conditions is,

$$\psi\left(x,y\right)=N\sin\left(k_x x\right)e^{ik_y y},$$

with energy (note we're only need the kinetic energy),

$$E=\frac{p^2}{2m}=\frac{\hbar^2\left(k_x^2+k_y^2\right)}{2m}.$$

Let us look at the wavefunction and the boundaries of the Möbius strip and work out what form the wave numbers will take.

Hard sides at $x=0$ and $x=W$, looking at the wavefunction's sine term, will set $k_x=\frac{n\pi}{W}$. The periodicity in the y-direction, is set by $k_y=\frac{m\pi}{L}$.

If m is odd, we get a negative sign $e^{ik_y L}=-1$. This will work if n is even: $\sin\frac{n\pi}{W}\left(W-x\right)=-\sin\frac{n\pi}{W}x$. If m is even, there is no minus sign, So n must be odd. Hence,

$$E=\frac{\hbar^2\pi^2}{2m}\left[\left(\frac{n}{W}\right)^2+\left(\frac{m}{L}\right)^2\right],$$

where $n=1,2,3,...$ and $m\,\epsilon\,\mathbb{Z}$, with $n+m$ being odd.

Now let us normalise the wavefunction, recall $\int\int\psi\left(x,y\right)\psi^*\left(x,y\right)\mathrm{d}x\,\mathrm{d}y=1$ and apply to our wavefunction $\psi\left(x,y\right)=N\sin\left(k_x x\right)e^{ik_y y}$.

Let's look at the two parts of the integral,

$$\left|e^{ik_y y}\right|^2=1$$

and for the $\sin^2\left(k_x x\right)$ term this averages to $\frac{1}{2}$ over one period. The easy way to see this:

$$\int_0^\pi\sin^2\left(\pi x\right)\mathrm{d}x=\int_0^\pi\cos^2\left(\pi x\right)\mathrm{d}x$$

and we can use,

$$\int_0^\pi\left(\sin^2\left(\pi x\right)+\cos^2\left(\pi x\right)\right)\mathrm{d}x=\pi.$$

Putting this together we have,

$$\int_0^L \int_0^W N^2 \sin^2\left(\frac{n\pi x}{W}\right) \mathrm{d}x \, \mathrm{d}y = N^2 \left(\frac{W}{2}\right) L = 1$$

and therefore, $N = \sqrt{\frac{2}{LW}}$. So, now we can put this all together and get the wavefunction

$$\psi(x, y) = \sqrt{\frac{2}{LW}} \sin\left(\frac{n\pi x}{W}\right) e^{\frac{im\pi}{L} y}.$$

Now, let us look at the time evolution of an electron confined on the Möbius strip.

$$\psi(t, x, y) = \sum c_{n,m} \, e^{\frac{-iE_{n,m}t}{\hbar}} \, \psi(x, y)$$

If $E_{n,m} = E_0 \times \mathbb{Z}$, then the time evolution is periodic, with period $T = \frac{\hbar}{E_0}$ (it doesn't matter what the initial wavefunction is). We now have

$$E_{n,m} = \frac{\hbar^2 \pi^2}{2m_e} \left[\left(\frac{n}{W}\right)^2 + \left(\frac{m}{L}\right)^2 \right]$$

where $L = 20$ nm and the width, $W = 3$ nm. Let us make $D = 60$ nm, then we have

$$E_{n,m} = \frac{\hbar^2 \pi^2}{2m_e D^2} \left[(3n)^2 + (20m)^2 \right],$$

here we have an integer always in the square bracket of the energy. Hence,

$$T = \frac{2m_e \hbar D^2}{\hbar^2 \pi^2} = \frac{2m_e D^2}{\pi^2 \hbar} = 3.15 \times 10^{-12} \, \mathrm{s}.$$

We can do a unit check here to make sure this makes sense:

$$\frac{\mathrm{kg \ m}^2}{\mathrm{kg \ m}^2 \mathrm{s}^{-2} \mathrm{s}} = s.$$

4 Icy Roof

After a night of frivolity, Patrick decides to climb on to the top of his roof. His house has an interesting shape - the upper half of a sphere of radius 10 m. It being an icy night, Patrick unsurprisingly starts to slip down.

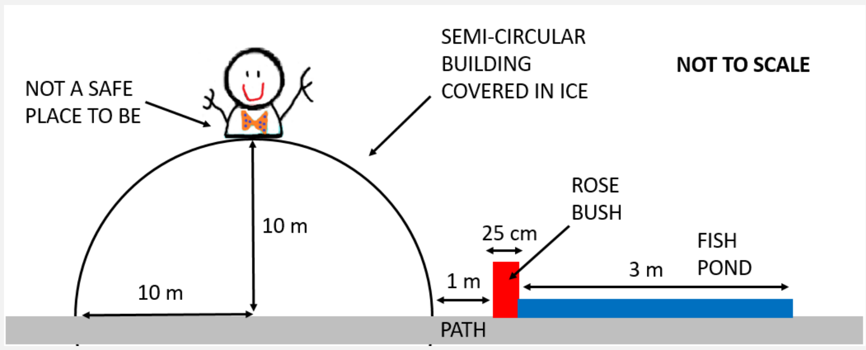


Figure 3: Sketch of Patrick's house and garden.

(a) [6 marks] Making the standard undergraduate physics approximations (no friction, point mass, etc.), calculate how far from the edge of the house Patrick lands, to see if he makes it into the pond, slams into the concrete path, or ends up in the thorny rose bush.

(b) [4 marks] Without detailed calculation (but with sound physics arguments), estimate how much of a difference it makes to relax the more unreasonable approximations to predict where Patrick will really land.

Icy Roof - Solution

(a) (6 marks) The first step is to figure out where the mass (Patrick) stops sliding along the roof and goes into free fall. Given that the roof is circular, one can do this in a relatively simple way [diagram to be added]. Call θ the angle between the centre of the circular roof and the mass, with $\theta = 0$ being the ground and $\theta = \pi/2$ being the top of the roof. Then the centripetal force required to keep the object moving in a circular path is v^2/r. This is provided by gravity – the component of gravity acting towards the centre of the circle is $g \sin\theta$. Hence the mass enters freefall when

$$\frac{v^2}{r} = g \sin\theta.$$

Now, by conservation of energy, $mgh + mv^2/2 = mgh_0$, where the height $h = r\sin\theta$. Hence

$$\frac{v^2}{2} = rg(1 - \sin\theta).$$

Putting these two equations together, the mass enters free-fall at an angle θ_0 given by

$$2(1 - \sin\theta_0) = \sin\theta_0,$$

so $3\sin\theta_0 = 2$ which gives $\sin\theta_0 = 2/3$.

We can now use our free-fall equations

$$\begin{aligned} x &= x_0 + v_{x0}t \\ y &= y_0 + v_{y0}t - gt^2/2. \end{aligned}$$

We have

$$\begin{aligned} x_0 &= r\cos\theta_0 = \frac{\sqrt{5}3}{r} \\ y_0 &= r\sin\theta_) = \frac{2}{3}r. \end{aligned}$$

Our initial velocity is given by the conservation of energy formula above,

$$v_0^2 = 2rg(1 - \sin\theta_0^2) = \frac{2rg}{3}$$

with components

$$\begin{aligned} v_{x0} &= v_0\sin\theta_0 = \sqrt{\frac{2rg}{3}}\frac{2}{3} \\ v_{y0} &= v_0\cos\theta_0 = \sqrt{\frac{2rg}{3}}\frac{\sqrt{5}}{3} \end{aligned}$$

With these initial conditions, we solve the free-fall equations to find the time when $y = 0$ (some more steps to be added but this is simple, just a bit messy):

$$t = \sqrt{\frac{r}{27g}}\left(\sqrt{46} - \sqrt{10}\right)$$

Substituting this into the x equation gives

$$x = r\left(\frac{\sqrt{5}}{3} + \frac{\sqrt{8}}{27}(\sqrt{46} - \sqrt{10})\right) = 1.1246r$$

Hence if r is $10m$, then it lands 1.24m from the edge of the building, which is in the rose bush.

(b) (4 marks) Remark in preparing exam – there are various alternative ways this question could be extended. One could look at more general initial conditions (i.e. not starting stationary at the top). You can also look at the condition for entering free-fall when the surface isn't a nice circle – this is a very nice calculation and has a beautiful answer. However, for PLANCKS, I opted for this more open-ended question allowing creativity and demonstration of ability to make reasonable estimates.

I will add some more full suggested answers, but given the open-ended nature, I expect the students to come up with better ideas than me. Marks will be given, not just for suggesting approximations to relax, but for a reasonable estimate of how much difference that will make. The main approximations I expect to be examined are

- Point mass approximation – To leading approximation, work with centre of gravity, which will be between $50-100$cm above the roof, depending on when he falls over. To a leading approximation, we could just replace r with a slightly larger value – which would mean he lands in the pond (at least his centre of gravity does...). For an extra mark, the students could also discuss him falling over as he slides down, along with rotational motion (which will change slightly the velocities involved)
- Friction - would increase the θ_0 where he enters freefall (as well as velocities of freefall). How would that change the position landed?
- Air resistance – probably small in this example, unless perhaps his coat is shaped as an aerofoil...

5 Floating Ring

A thin superconducting ring is held above a vertical, cylindrical magnetic rod. The axis of symmetry of the ring is the same as that of the rod. The cylindrically symmetrical magnetic field around the ring can be described approximately in terms of the vertical and radial components of the magnetic field vector as $B_z = B_0\,(1 - \alpha z)$ and $B_r = B_0 \beta r$, where B_0, α and β are constants, and z and r are the vertical and radial position coordinates, respectively.

A sketch of this can be seen in Figure 4.

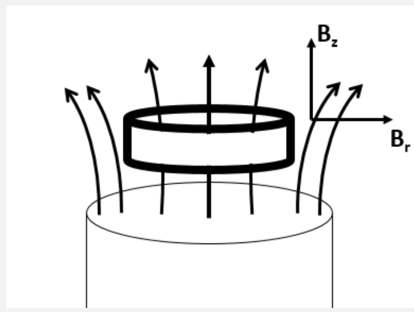


Figure 4: A thin superconducting ring above a cylindrical metal rod.

Initially, the ring has no current flowing in it. The coordinates of the centre of the ring are $(z, r) = (0, 0)$. When released, it starts to move downwards with its axis still vertical.

Useful data:

- Ring's mass, $m = 50$ mg
- Ring's radius, $r_0 = 0.5$ cm
- Ring's inductance, $L = 1.3 \times 10^{-8}$ H
- Magnetic field constant, $B_0 = 0.01$ T
- Magnetic field constant, $\alpha = 2$ m^{-1}
- Magnetic field constant, $\beta = 32$ m^{-1}

(a) [8 marks] Show that the ring undergoes simple harmonic motion, and find the frequency and amplitude of the oscillation.

(b) [2 marks] What is the maximum current that flows in the ring, and where in the oscillation does it occur?

Floating Ring - Solution

In this problem we apply the laws of inductance and the Lorentz force to the senario of a thin superconducting ring. We do not require extensive knowledge of superconductors other than they conduct current really well.

The total magnetic flux at the position of the ring is made up of contributions from the external magnetic and the effects of self-inductance,

$$\Phi = B_z \pi r_0^2 + LI.$$

The change in the magnetic flux will induce a current in the ring according to

$$RI = \frac{\Delta\Phi}{\Delta t}.$$

In a superconductor, Ohmic resistance, $R = 0$ and therefore we can state that the total magnetic flux through the ring has to be a constant,

$$\Phi = B_z \pi r_0^2 + LI = B_0 \left(1 - \alpha z\right) \pi r_0^2 + LI = \text{constant}.$$

Apply the initial conditions $(z = 0, I = 0)$ and we find that the constant value is $\Phi = B_0 \pi r_0^2$. By substituting and rearranging the above equations, we find that the current is

$$I\left(z\right) = \frac{1}{L} B_0 \alpha \pi r_0^2 z.$$

Only the vertical component of the Lorentz force acts on the ring due to the symmetry of the setup, we can express this force using $F = IlB \sin\theta$,

$$F_z = -B_r I\left(z\right) 2\pi r_0 = \frac{-B_0 \beta r_0 \alpha \pi r_0^2 z}{L} = -kz.$$

The above result shows that the Lorentz force is directly proportional to the vertical displacement of the ring. We can calculate the constant of proportionality using the information given in the question. Note: this is only valid for small displacements.

We can now write the equations of motion for the ring, please note that if the force moving the ring up is the Lorentz force then the force bringing it down is gravity, and you may assume we are on the Earth.

The equation of motion of the ring is

$$ma_z = F_z - mg = -kz - mg.$$

We can see that the ring makes harmonic oscillations about the equilibrium position $z_0 = \frac{-mg}{k}$ with $z\left(z\right) - z_0 = A\cos\left(\omega t\right)$, where $\omega = \sqrt{\frac{k}{m}}$. From the initial conditions we can find $A = -z_0$ and therefore

$$z\left(t\right) = \frac{g}{\omega^2}\left(\cos\left(\omega t\right) - 1\right).$$

The vertical z-coordinate is never positive, and it follows that the Lorentz force always points upwards, being zero at the topmost point of the oscillation. The current always flows in the same direction around the ring.

Finally, we can now substitute to find the numerical answers to the problem.

Frequency:

$$\omega = \sqrt{\frac{2\pi^2 r_0^4 B_0^2 \alpha\beta}{Lm}} = 11.02\,\text{rads}^{-1}.$$

or

$$f = \frac{\omega}{2\pi} = 1.75\,\text{Hz}.$$

Amplitude:

$$A = \frac{g}{\omega^2} = \frac{Lmg}{2\pi^2 r_0^4 B_0^2 \alpha\beta} = 8.1\,\text{cm}.$$

Current:

$$I_{max} = \frac{\pi r_0^2 B_0 \alpha}{L} \times 2A = 19.6\,\text{A}.$$

At the bottom of the oscillation is where the max current flows.

It is worth noting that the magnetic fields in the question do not satisfy Maxwell's equations, well done to the participants who spotted this.

6 Nuclear Medicine Scan

Red deer are native to Ireland, as they get older the male deer grow very large antlers. A young deer wanted to check if his budding antlers were growing uniformly. So, the Stag visited St James' Hospital in Dublin. He received a nuclear medicine scan, whereby he was injected with a photon-emitting radionuclide which accumulates primarily, but not exclusively, in areas of bone growth (his antlers).

The signal from the radioactivity measured from a detector placed outside an object will depend on both the activity present and on the distribution of the photon attenuation coefficients.

Consider an object with a distribution of attenuation coefficients $\mu(x, y)$ and radioactivity distribution $A(x, y)$ as shown in Figure 5.

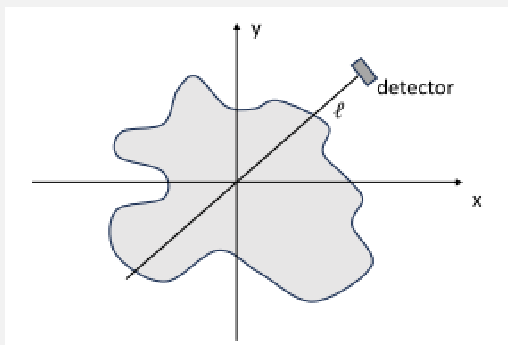


Figure 5: Object with arbitrary activity distribution $A(x, y)$ and attenuation coefficient distribution $\mu(x, y)$.

Assume that the detector has 100% efficiency (i.e., all photons hitting it are detected) and that it is collimated (i.e., it will only detect photons emitted along a thin line). The photons emitted, from a point source, in a specific direction are proportional, by a factor $k < 1$, to the total number of photons emitted.

Remember that the variation in intensity for a beam travelling through an infinitesimally small thickness $\mathrm{d}t$ is

$$\mathrm{d}I = -\mu I \mathrm{d}t.$$

(a) [1 mark] In the absence of attenuation, the signal measured along an arbitrary line ℓ is given by

$$I = k \int_0^{\ell} A(x, y)\, \mathrm{d}s.$$

How would this be modified when attenuation is taken into account? Assume that there is no geometrical correction for the distance of the emission point from the detector.

Consider now the top of the Stag's skull, approximated by an ellipse with semi-axes a and b, arbitrary radioactivity distribution $A(R, \vartheta)$ in polar coordinates and a constant attenuation coefficient µ (see Figure 6).

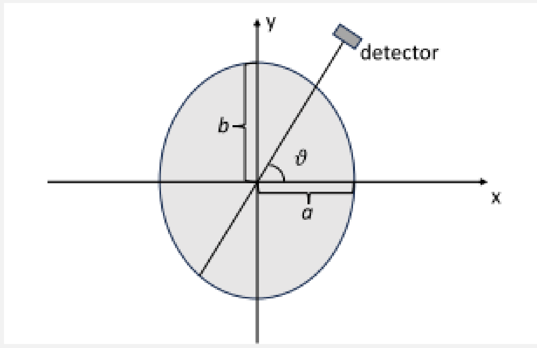


Figure 6: Ellipse with arbitrary activity distribution and constant attenuation coefficient.

(b) [3 marks] Write an expression, in polar coordinates, for the signal measured from a line going through the origin of the axes at an angle ϑ to the x-axis.

So far we have assumed a point-like detector with infinite angular resolution (i.e., it will only detect photons emitted along a very thin line). The detector now has a circular aperture of radius R, as shown in Figure 7).

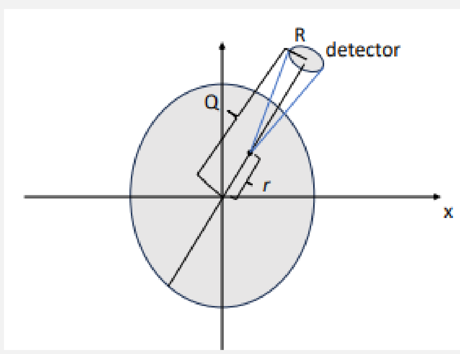


Figure 7: Activity distribution in a plane, with a circular detector aperture

We can assume that the radioactivity is still distributed in a plane, but that photons are emitted isotropically in a sphere.

(c) [3 marks] How would your solution to part b change if we had a detector with a circular aperture of radius R?

The diagram in Figure 8 shows the Stag's scan, with a diffuse "background" uptake of radionuclide and two "points" representing the antler buds.

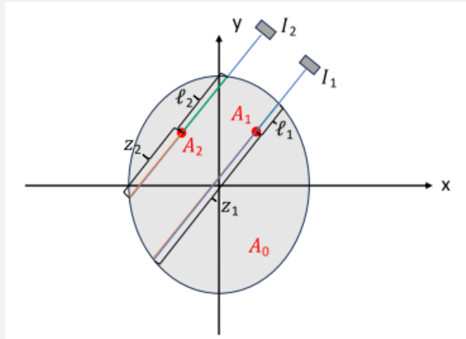


Figure 8: Diagram of the Stag's scan.

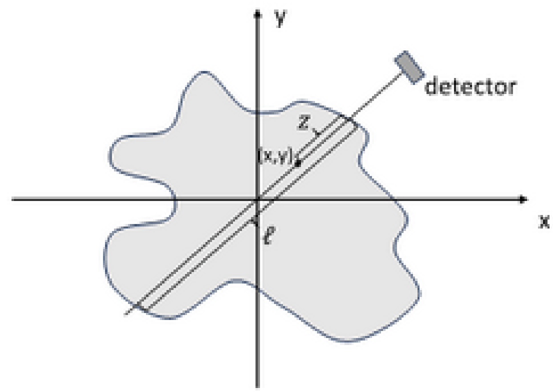
- The detector counts measured along the two parallel lines are $I_1 = 1.23\ \text{s}^{-1}$ and $I_2 = 1.15\ \text{s}^{-1}$.
- The two antler buds can be treated as point-like, with radioactivity uptakes A_1 and A_2. If $A_1 = A_2$, the antlers were growing uniformly and the Stag will now have developed a pair of symmetrical antlers.
- The radioactivity/unit length in the rest of the skull is $A_0 = 3000\ \text{Bqcm}^{-1}$.
- The attenuation coefficient of the head tissue is $0.27\ \text{cm}^{-1}$.
- $\ell_1 = 5.25$ cm, $\ell_2 = 11$ cm, $z_1 = 24.5$ cm and $z_2 = 12.5$ cm
- For this part of the question, assume again a perfectly collimated detector, with the ratio between photons emitted in the detector's direction/total photons emitted at each point $k = 10^{-4}$.

(d) [3 marks] On the basis of the quantities above, calculate A_1 and A_2 and determine if the Stag's antlers were growing uniformly at the time of the scan.

Nuclear Medicine Scan - Solution

(a) [1 mark]

When attenuation is not negligible, we have to take into account the depth z of each emission point from with respect to the object's surface closest to the detector along the line of emission:



$$I = k \int_0^{\ell} A\left(x, y\right) e^{-\int_0^z \mu(x', y') \mathrm{d}s'} \mathrm{d}s.$$

(b) [3 marks] Properties of ellipses:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

$$\begin{cases} X = a \cos \vartheta \\ Y = b \sin \vartheta \end{cases}$$

where (X, Y) is a point on the ellipse.

The signal I on the detector will be proportional, through the same constant k as before, to the line integral of all the activities along the line ℓ, each corrected by the attenuation along the segment s between the point of emission and the closest point (X, Y) to the detector along ℓ.

Let's take a point at position (x, y). There are two cases, the first is where (x, y) is in the opposite quadrant as the detector and the second case is where (x, y) is in the same quadrant as the detector.

Using we can find an expression for s in these two cases, written in polar coordinates:

In case 1, $s = r + \sqrt{a^2 \cos^2 \vartheta + b^2 \sin^2 \vartheta}$ and in case 2, $s = \sqrt{a^2 \cos^2 \vartheta + b^2 \sin^2 \vartheta} - r$.

So, our expression will be

$$I = k\left[\int A\left(r,\vartheta\right)e^{-\mu s}\mathrm{d}s\right]_{\text{opposite quadrant}} + k\left[\int A\left(r,\vartheta\right)e^{-\mu s}\mathrm{d}s\right]_{\text{same quadrant}},$$

which we can then rewrite by substituting in the expressions for s in the two cases and then writing the integrals with respect to the radius.

(c) [3 marks] If we include the angular acceptance of the detector, we need to factor in the distance between each emission point and the detector, d, and the radius of the detector, R.

The fraction of emitted photons hitting the detector is

$$\mathcal{F} = \frac{\pi R^2}{4\pi d^2} = \frac{R^2}{4d^2}.$$

But for a generic point at a radial position r, if Q is the distance between the detector and the origin,

$$\begin{cases} d = Q - r & \text{for points in the same quadrant as the detector} \\ d = Q + r & \text{for points in the opposite quadrant as the detector} \end{cases}$$

You can then rewrite our expression found in the last part of the question with the expressions we've found for the distance between each emission point.

(d) [3 marks] The signal measured by the detector in each of the two positions is proportional to the line integral of the activity/unit length along the corresponding line of response, at each point attenuated by the underlying tissue, plus the activity in the antler bud attenuated by the underlying tissue:

$$I_{1,2} = k\left[\int_0^{z_{1,2}+\ell_{1,2}} A_0 e^{-\mu x}\mathrm{d}x + A_{1,2}e^{-\mu\ell_{1,2}}\right].$$

After evaluating the integral, the total signal measured in each of the two positions is given by

$$I_{1,2} = k\left[\frac{A_0}{\mu}\left(1 - e^{-\mu(z_{1,2}+\ell_{1,2})}\right) + A_{1,2}e^{-\mu\ell_{1,2}}\right].$$

Rearrange for $A_{1,2}$ and input the quantities given, we obtain $A_1 = 4921$ Bq and $A_2 = 7960$ Bq. The ratio of these is 61.8% and therefore the antlers were not growing uniformly.

7 Dark Matter in the Galaxy

In a galaxy all objects inside orbit around its centre. At a point $r = 2.6 \times 10^4$ ly from the centre the measured velocity of orbit is $v_{meas} = 250\ \text{kms}^{-1}$. However, if we were to calculate the velocity of orbit at distance r based on all the luminous matter in the galaxy, the value would be much lower, $v_{calc} = 15\ \text{kms}^{-1}$. Why is there such a large difference? It can only be possible if the galaxy had a lot more matter inside but it was hidden from view. The hidden matter is now called dark matter and we know it exists throughout the universe, even though we have no idea what it is.

Imagine you are an astronomer in this galaxy who is investigating dark matter in the galaxy's spherical halo.

Useful data:

- Gravitational constant, $G = 6.67 \times 10^{-11}\ \text{Nm}^2\text{kg}^{-2}$
- One light year, $1\ \text{ly} = 9.46 \times 10^{15}$ m
- One electron volt, $1\ \text{eV} = 1.602 \times 10^{-19}$ J

(a) [5 marks] Calculate the average dark matter density in this region of the galaxy. Give your answer in $\frac{\text{GeV}}{\text{c}^2}\text{cm}^{-3}$.

Your galaxy is in a universe where where all the dark matter is made up of miniature black holes. These mini black holes are spread throughout a galaxy's spherical halo (like a dark matter halo).

The mini black holes orbit around the centre of the galaxy like every other object. Your planet sits approximately at distance r from the centre of the galaxy. Through observation you notice that roughly once a year one of these mini black holes pass between you and your local star which you can assume is at distance ~ 1 AU to you.

Useful data:

- One astronomical unit, $1\ \text{AU} = 1.50 \times 10^{11}$ m
- Solar mass, $1\ \text{M}_\odot = 2 \times 10^{30}$ kg

(b) [5 marks] What is the approximate mass of these miniature black holes? Give your answer in units of the Solar mass.

State any assumptions you have made to get to your answers.

Dark Matter in the Galaxy - Solution

(a) [5 marks] For the first part we must calculate the dark matter density and have been given an orbital radius and speed. This is a fairly straightforward application of Kepler's laws, the important part was remembering to subtract the visible mass density from the total.

Our formula for the dark matter density is given by:

$$\rho_{DM} = \frac{M_{DM}}{V_{Region}} = \frac{M_{Measured} - M_{Calculated}}{V_{Region}}$$

We can rewrite the mass in terms of velocity using $v(R) = \sqrt{\frac{GM(R)}{R}} \to M(R) = \frac{v^2(R)R}{G}$, which yields

$$\rho_{DM} = \frac{1}{V_{Region}} \left(\frac{r}{G}\right) \left(v^2_{Measured} - v^2_{Calculated}\right).$$

If we assume the volume of the region is spherical we arrive at our final expression for the density:

$$\rho_{DM} = \frac{3}{4\pi r^2 G} \left(v^2_{Measured} - v^2_{Calculated}\right).$$

Now, we plug in the values given in the question, but don't forget to convert units ($1\text{kg} = 5.61 \times 10^{26} \frac{\text{GeV}}{\text{c}^2}$). This gives our dark matter density in the region to be $2.07 \frac{\text{GeV}}{\text{c}^2}\text{cm}^{-3}$

(b) [5 marks] For this part of the question, you have to decide what does it mean for a black hole to fly between the Earth and the Sun. Participants gave some excellent answers with in-depth reasoning and discussion behind their choices. We opted for the simplest of assumptions- a box of 1 Au $\times V_{Orbit} \times$ 1 year. The mass of the black holes can be found by simply taking the total mass of the region and dividing it by the number of black holes in that region,

$$M_{BH} = \frac{M_{Region}}{\text{Number of black holes}} = M_{Region} \times \frac{V_{unit}}{V_{Region}}.$$

From the first part we calculated $M_{Region} = 2.31 \times 10^{41}$ kg and $V_{Region} = 6.23 \times 10^{61}$ m^3.

Let us consider the volume that one black hole would occupy.

The average distance from the closest object inside a cube of a given volume, would be approximately half the length of the longest diagonal inside the cube ($2d = \sqrt{3} \times$ side of cube). The time taken for a black hole to cover the distance up to 1 Au from the Sun would be of order $t \sim \frac{d}{v_{Orbital}}$. We can use these to approximate the unit volume occupied by a black hole

$$V_{unit} = \left(\frac{2d}{\sqrt{3}}\right)^3 \simeq \left(\frac{2 \times t_{1year} \times v_{Orbit}}{\sqrt{3}}\right)^3.$$

We find with the box approximation $M_{BH} \sim 10^{-12}\text{M}_\odot$, acceptable masses ranged from $10^{-8}\text{M}_\odot$ to $10^{-14}\text{M}_\odot$ depending on what participants decided on what it means to pass between the Earth and Sun.

8 Icosahedron of Resistors

An icosahedron consists of 20 equilateral triangles. It has 12 vertices and 30 edges, with 5 edges meeting at each vertex. Figure 9 shows an icosahedron and its net.

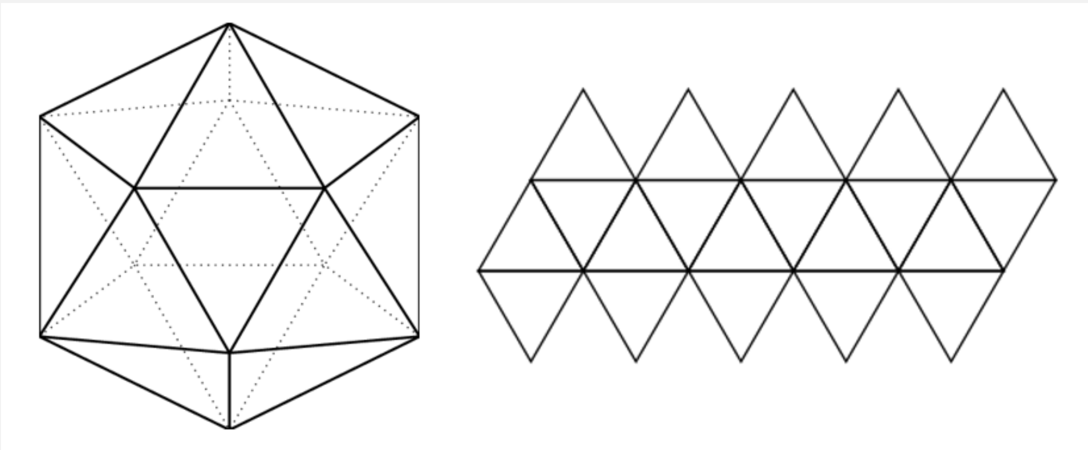


Figure 9: Icosahedron (left) and its 2D net (right).

Now imagine that this icosahedron was a component in a circuit, where each edge of the icosahedron is a 1Ω resistor.

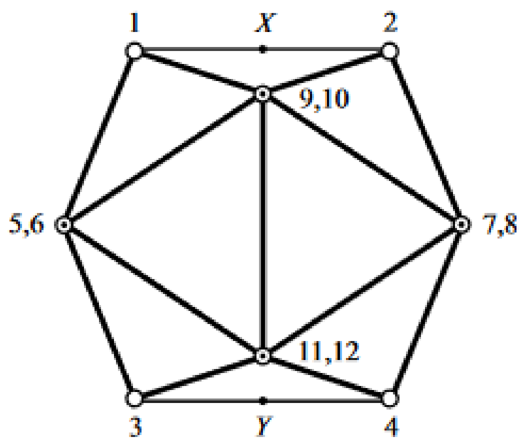
You may use the *model icosahedron kit* provided to you.

(a) [10 marks] Find the effective resistance between two adjacent vertices.

Icosahedron of Resistors - Solution

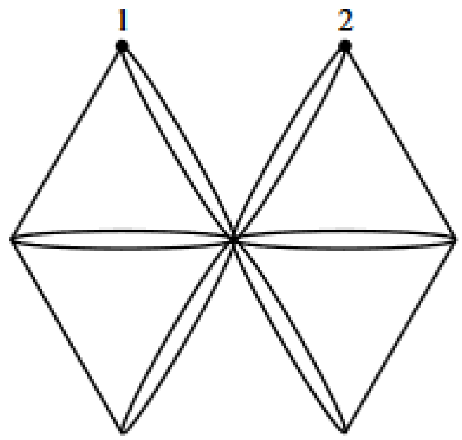
The author presents two solutions to the problem.

First Solution: We will calculate the effective resistance between vertices 1 and 2 in the figure below. When the icosahedron is viewed from the angle shown, four vertices lie directly behind four other vertices; each of these pairs is represented by a dot inside a circle. And thirteen edges lie directly behind thirteen other edges; each of these pairs is represented by a bold line. The remaining two (of the 30 total) edges not represented in the figure are the ones connecting vertices 5 and 6, and 7 and 8. For future reference, points X and Y are defined to be the midpoints of the edges shown.

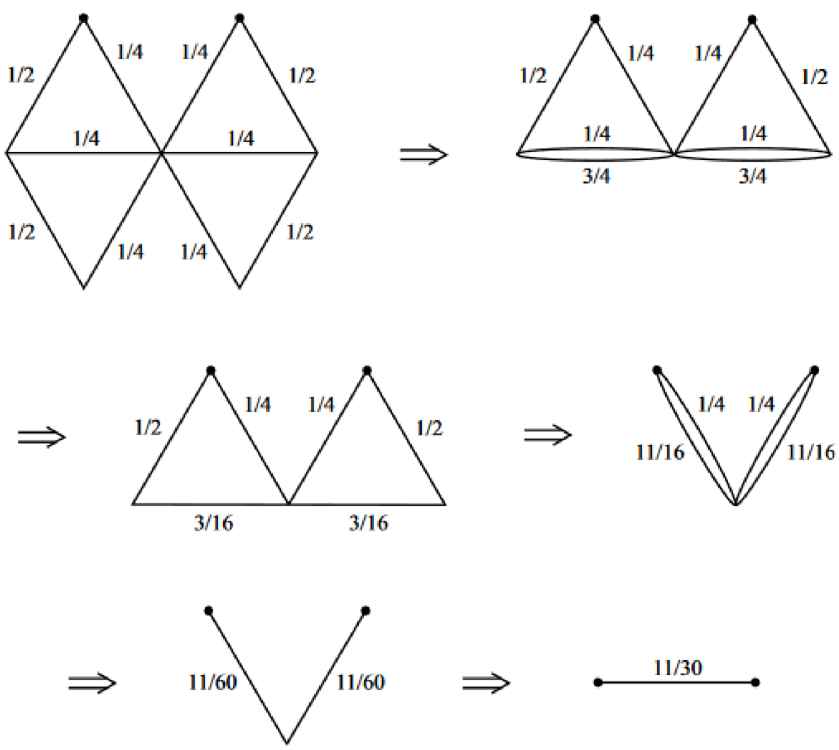


If a potential difference is created between vertices 1 and 2, then the potentials at vertices 5 and 6 are equal, and likewise for the other three pairs of vertices. We may therefore bring each of these four pairs of points together and identify each pair as one point. The resulting circuit is simply the figure above, now planar, where each bold line represents a $\frac{1}{2}\Omega$ resistor (because it arises from two 1Ω resistors in parallel).

We now note that all points on the vertical bisector of the circuit (that is, X, Y, the 9-10 pair, and the 11-12 pair) are at equal potentials, so we may bring them all together and identify them as one point. Since X and Y split the top and bottom resistors into two $\frac{1}{2}\Omega$ resistors, we arrive at the following circuit, where every line in the figure represents a $\frac{1}{2}\Omega$ resistor.



This circuit may be reduced as follows:



The effective resistance between two adjacent vertices is therefore $\frac{11}{30}\Omega$.

Second Solution: Let vertices 1 and 2 be adjacent. Consider the setup where a current 1A enters through vertex 1, and a current $\frac{1}{11}$A leaves through the other 11 vertices. Note that, due to symmetry, a current $\frac{1}{5}$A flows through each of the 5 edges leaving vertex 1. Hence, the voltage difference between vertices 1 and 2 is

$$V_1 - V_2 = \left(\frac{1}{5}\text{A}\right)(1\Omega) = \frac{1}{5}\text{V}.$$

Consider a second setup, where a current 1A leaves through vertex 2, and a current $\frac{1}{11}$A enters through the other 11 vertices. Again, note that a current $\frac{1}{5}$A flows through each of the 5 edges entering vertex 2. Hence, the voltage difference between vertices 1 and 2 is

$$V_1 - V_2 = \left(\frac{1}{5}\text{A}\right)(1\Omega) = \frac{1}{5}\text{V}.$$

If we superimpose these two setups on each other, then we arrive at the setup where:

- A current $\frac{12}{11}$A enters through vertex 1,
- A current $\frac{12}{11}$A leaves through vertex 2,
- No current enters or leaves through the other 10 vertices, and
- The potential difference between vertices 1 and 2 is

$$V_1 - V_2 = \frac{1}{5}\text{V} + \frac{1}{5}\text{V} = \frac{2}{5}\text{V}.$$

We have therefore constructed precisely the experimental setup that serves to determine the effective resistance between vertices 1 and 2. That is, we have put a current in at vertex 1, taken it out at vertex 2, and measured the voltage difference between the two points. The effective resistance between vertices 1 and 2 is therefore given by

$$V = IR \Rightarrow \frac{2}{5}\text{V} = \left(\frac{12}{11}\text{A}\right) R_{eff} \Rightarrow R_{eff} = \frac{11}{30}\Omega.$$

9 Molecular Zipper

In the 1960s, Charles Kittel proposed a toy model to illustrate the physics of the separation of strands of DNA. The model consists of two long molecules coupled by N links, with the following rules:

- Each link can be closed or open – and if open, they can be in one of G orientations where $G > 0$ is an integer parameter of the model. There is only one way the link can be closed.
- If links 1 to n are open, then link $n + 1$ can also open with energy cost $\epsilon > 0$. This is the basic idea of the zipper model – a link can only be open if all the ones before it are also open.
- The final link $n = N$ can not be opened.

The final rule means that the model is a 'single-ended zipper' – which is a slight simplification of earlier models that could be 'unzipped' from both ends.

(a) [4 marks] Calculate the free energy of the model with N links at a temperature T.

(b) [2 marks] Hence show that the model has a finite-temperature phase transition if $G > 1$, and find an expression for the transition temperature.

(c) [2 marks] Calculate the expected number of open links as a function of temperature, and show that this may be used as an order parameter for the phase transition.

(d) [2 marks] Come up with a physical explanation why one requires this extra degeneracy $G > 1$ in order to have a phase transition in the model.

Molecular Zipper - Solution

Reference paper for further reading: C. Kittel, Phase transition of a molecular zipper, Am. J. Phys. 37, 917 (1969).

(a) (4 marks) The free energy F at a temperature T is given by

$$F = -k_B T \ln Z$$

where Z is the partition function

$$Z = \sum_{\text{all states}} e^{-E/k_B T}.$$

Here, k_B is Boltzmann's constant.

In this case, the state of the system is given by the number of links open p which can run from $p = 0$ to $p = N - 1$ (the N'th link cannot be opened. The energy of such a configuration is simply $p\epsilon$ where ϵ is the energy cost of a link being opened. In addition, for each open link, there are G possible states. These don't affect the energy, so can be taken into account via a degeneracy of the state of G^p. Hence

$$Z = \sum_{p=0}^{N-1} G^p \exp(-p\epsilon/k_B T).$$

If we write

$$x = Gexp{-}\epsilon/k_B T,$$

then

$$Z = \sum_{p=0}^{N-1} x^p = \frac{1 - x^N}{1 - x}$$

where the expression for the sum of a geometric series is easily derived if not recalled.

Putting this together,

$$F = -k_B T \ln\left(\frac{1 - x^N}{1 - x}\right), \quad x = Gexp{-}\epsilon/k_B T.$$

(b) (2 marks) The expression for free energy can have a singularity if $x = 1$.[4] Setting

$$G \exp(-\epsilon/k_B T_c) = 1$$

gives us

$$T_c = \frac{\epsilon}{k_B \ln G}.$$

If $G = 1$, then $\ln_G = 0$ and hence $T_c \to \infty$, however for any $G > 1$, this is a finite transition temperature.

[4] If one wants to be more specific, there is no singularity for finite N – the sum is analytic. However in the thermodynamic limit $N \to \infty$, this would be a transition between an intensive free energy (doesn't scale with system size) for $x < 1$ to an extensive free energy for $x > 1$. Hence a phase transition.

(c) (2 marks)

The probability of being in each state is the Boltzmann weight $e^{-E/k_B T}$, hence the expected number of open links

$$\langle p\rangle = \frac{1}{Z}\sum_{p=0}^{N-1} p x^p$$

where we have defined x as before.

Either one can memorise sums like this, or know how to derive them when needed. In this case it is easy to derive:

$$\sum_{p=0}^{N-1} p x^p = x\frac{d}{dx}\sum_{p=0}^{N-1} x^p = x\frac{dZ}{dx}.$$

Hence

$$\begin{aligned}\langle p\rangle &= x\frac{1-x}{1-x^N}\left(\frac{-Nx^{N-1}}{1-x} + \frac{1-x^N}{(1-x)^2}\right)\\ &= \frac{x}{1-x} - \frac{Nx^N}{1-x^N}.\end{aligned}$$

For large N (the thermodynamic limit), the expression is dominated by the second term for $x > 1$, giving N open links – i.e. completely unzipped. For $x < 1$ however, the second term goes to zero (in the thermodynamic) limit, leaving only the first term which is not proportional to N. Hence we see:

- For $x < 1$ which is the low temperature phase, $T < T_c$, the system is zipped, with on average a non-extensive number of links unzipped.
- For $x > 1$ which is the high temperature phase $T > T_c$, the system is completely unzipped.

In his paper, Kittel does a further analysis, examining $\langle p\rangle$ in the vicinity of the phase transition $x = 1 + \eta$ where $\eta \ll 1$. He looks at the fraction of links unzpped $\langle p\rangle/N$ as a function of T and shows the slope becomes infinite at T_c in the thermodynamic limit, providing more characterisation of the phase transition. See his paper for more details if you are interested.

(d) (2 marks) It is not a-priori obvious why this zipper model requires a degeneracy $G > 1$ in order to have a phase transition.[5]. Knowing that it is a zipping/unzipping transition though allows one to construct a very nice explanation.

A thermodynamic phase transition (such as ferromagnetism) is usually due to a competition between energy and entropy – the low tempeature 'ordered' phase lowers the energy at the expense of energy, while the high temperature paramagnetic 'disordered' phase has a high entropy at the cost of not being a low energy state. One can see mathematically why this is the case from the expression $F = E - TS$ – at low temperature the entropy isn't important so one minimises energy; at sufficiently high temperature, the entropy will dominate.

The zipper model is the same. It costs energy to unzip, so at low temperature we minimise energy by being zipped. Entropy can only be released by unzipping though, if there are more

[5] And the more you know about phase transitions the harder it gets – e.g. phase transitions aren't supposed to happen at all in one dimension. For a very nice discussion of what makes this model an exception to this rule, see Cuesta and Sanchez, J. Stat. Phys. 115 , 869 (2004).

'unzipped' states than 'zipped'. This happens only if the degeneracy of the unzipped states is two or more – as we saw in the calculation.

Partial marks will be given for any sound physical arguments that differ from the above, however the competition between energy and entropy must be mentioned for full marks.

10 4D Sun

Imagine you were transported to another universe with one more spatial dimension, everything appeared as 4D. In this problem, you will investigate if the light generated from the Sun would be different in this universe.

You may assume that the surface temperature of the Sun would not change from our universe and is approximately $T = 6 \times 10^3$ K.

Useful data:

- Boltzmann constant, $k_B = 1.38 \times 10^{-23}$ JK^{-1}
- Speed of light, $c = 3 \times 10^8$ ms^{-1}

(a) [10 marks] What would the colour of the Sun be in this four-dimensional universe?

4D Sun - Solution

Black body radiation, spectral density:

$$\varepsilon(\nu)\,\mathrm{d}\nu = h\nu\rho(\nu)\,n(\nu)$$

The photon energy, $E = h\nu$ where h is Planck's constant and ν is the photon frequency.

The density of states, $\rho(\nu) = A\nu^{n-1}$ where, A is a constant independent of the frequency and frequency term is the scaling of surface area of an n-dimensional sphere.

The Bose-Einstein distribution, $n(\nu) = \frac{1}{e^{\frac{h\nu}{kT}}-1}$ where k is the Boltzmann constant and T is the temperature.

We let $x = \frac{h\nu}{kT}$ and get $\varepsilon(x) \propto \frac{x^n}{e^x-1}$.

We do not need the constant of proportionality (which is not simple to calculate in 4D) to find the maximum of $\varepsilon(x)$. Working out the constant just tells us how tall the peak it, but we are interested in where the peak is, not the total radiation.

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}x} \propto \frac{nx^{n-1}}{e^x-1} - \frac{x^n e^x}{(e^x-1)^2}$$

We set equal to zero for the maximum of the distribution,

$$\frac{x^{n-1}e^x}{(e^x-1)^2}\left(n\left(1-e^{-x}\right)-x\right) = 0$$

This yields $x = n(1-e^{-x})$ where, $x = \frac{h\nu_{max}}{kT}$ and we can relate $\lambda_{max} = \frac{c}{\nu_{max}}$ and c being the speed of light.

This equation has the solution $x = n + W(-ne^{-n})$ where W is the Lambert W function $z = W(y)$ that solves $ze^z = y$ (although there is a subtlety about which branch of the function). This is kind of useless to do anything with, though. One can numerically solve this equation using bisection/Newton-Raphson/iteration. Alternatively, one could notice that as the number of dimensions increases, e^{-x} is small, so to leading approximation $x \approx n$. One can do a little better iterating this, $x \approx n - ne^{-n}$ which is what we will use. Note the second iteration yield $x \approx n - ne^{\left(n-ne^{-n}\right)}$.

Number of dimensions, n	Numerical solution	Approximation using first iterating
2	1.594	1.729
3	2.821	2.851
4 (the one we want)	3.921	3.927
5	4.965	4.966
6	5.985	5.985

Using the result above,

$$\lambda_{max} = \frac{hc}{kTx_{max}} = \frac{6.63 \times 10^{-34} \cdot 3 \times 10^{8}}{1.38 \times 10^{-23} \cdot 6 \times 10^{3} \cdot 3.9} = 616\,\mathrm{nm}$$

616 nm is middle of the spectrum, so it will look white with a green-blue tint. Note, we have used $T = 6000$ K for the temperature here, as given in the question.

It would also be valid to look at $\varepsilon\left(\lambda\right)\mathrm{d}\lambda$ instead of $\varepsilon\left(\nu\right)\mathrm{d}\nu$.

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